SOLVING ORDINARY DIFFERENTIAL EQUATIONS USING POWER SERIES

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Abstract: In this work, we studied that Power Series Method is the standard basic method for solving linear differential equations with variable coefficients. The solutions usually take the form of power series; this explains the name Power series method. We review some special second order ordinary differential equations. Power series Method is described at ordinary points as well as at singular points (which can be removed called Frobenius Method) of differential equations. We present a few examples on this method by solving special second order ordinary differential equations.

Key Words: Power series, differential equations, Frobenius Method, Lengendre polynomials.

1.0 INTRODUCTION:

1.1 BACKGROUND OF THE STUDY:

The attempt to solve physical problems led gradually to Mathematical models involving an equation in which a function and its derivatives play important role. However, the theoretical development of this new branch of Mathematics –Differential Equations— has its origin rooted in a small number of Mathematical problems. These problems and their solutions led to an independent discipline with the solution of such equations an end in itself (Sasser, 2005).

1.2 STATEMENT OF THE PROBLEM:

The research work seeks to find solutions of second-order ordinary differential equations using the power series method.

1.3 AIM AND OBJECTIVES:

The aim and objectives of the study are to:

I. Describe the power series method.
II. Use it to solve linear ordinary differential equations with polynomial coefficients.
III. To review some special second order ordinary differential equations.
IV. To determine the solutions of these special ordinary differential equations by the power series solution method.

1.4 SCOPE OF THE STUDY:

There are other classes of ordinary differential equations but we will be concerned with second-order linear ordinary differential equations especially those with variable coefficients (polynomial coefficients), with emphasis on special ordinary differential equations such as Bessel, Airy, Legendre and Hermite equations.

Moreover, there are several methods of solving the differential equations but we will be concerned with the series solution method.

1.5 JUSTIFICATION OF THE STUDY:

Most ordinary differential equations are cumbersome and complex, and cannot be solved by exact or elementary methods analytically especially when adequate information such as graphs is not supplied. Their solutions can only be approximated using Numerical methods with appropriate boundary or initial conditions. Using power series method however, is a more systematic way and standard basic method for approximating the solutions of such differential equations analytically and thus studying the method is of greater importance.

1.6.0 BASIC CONCEPTS AND DEFINITIONS:

1.6.1 POWER SERIES:

A series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots \tag{1.1}$$

is called power series in power of $(x - x_o)$. Where a_n's are coefficients of the power series usually constants. The point x_0 is called the center of the power series and x a variable.

The term "power series" alone usually refers to a series of the form (1.1), but does not include series of negative powers or series involving fractional powers of $(x - x_0)$. For convenience, we write $(x - x_0)^0 = 1$, even when $x = x_0$.

1.6.2 CONVERGENCE OF POWER SERIES:

We say that (1.1) converges at the point $x = c$ if the infinite series (of real numbers)

$$\sum_{n=0}^{\infty} a_n(c-x_0)^n$$

Converges; that is, the limit of the partial sums,

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(c-x_0)^n$$

exists (as a finite number). If this limit does not exist, the power series is said to diverge at $x = c$. We can observe that (1.1) converges at $x = x_0$, since

$$\sum_{n=0}^{\infty} a_n(x_0-x_0)^n = a_0 + 0 + 0 + \cdots = a_0.$$

A power series of the form (1.1) converges for all values of x in some "interval" centered at x_0 and diverges for x outside this interval. Moreover, at the interior points of this interval, the power series converges absolutely in the sense that

$$\sum_{n=0}^{\infty} |a_0(x-x_o)^n|$$

Converges.

2.0 LITERATURE REVIEW:

2.1 ORDINARY DIFFERENTIAL EQUATIONS:

Ince (1956) observed that the study of differential equations began in 1675 when Leibniz wrote the equation:

$$\int x dx = (1/2)x^2.$$

Leibniz inaugurated the differential sign $(\frac{dy}{dx})$ and integral sign $(\int)$ in (1675) a hundred years before the period of initial discovery of general methods of integrating ordinary differential equation ended.

According to Sasser (2005) the search for general methods of integrating began when Newton classified the first order differential equations into three classes;

$$\frac{dy}{dx} = f(x) \tag{2.0}$$

$$\frac{dy}{dx} = f(x, y) \tag{2.1}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u \tag{2.2}$$

The first two classes contain only ordinary derivatives of one or more dependent variables, with respect to a single independent variable, and are known today as ordinary differential equations. The third class involved the partial derivatives of one dependent variable and today is called partial differential equations.

Billingham and King (2003) studied mathematical modeling and outline the relevance of Ordinary differential equation in modeling dynamic systems. Saying, it gives the conceptual skills to formulate, develop, solve, evaluate, and validate such systems. Many physical, chemical and biological systems can be described using mathematical models. Once the model is formulated, we usually need to solve a differential equation in order to predict and quantify the features of the system being modelled.

3.0 RESEARCH METHODOLOGY:

In this chapter we study power series and discuss the use of power series to construct fundamental sets of solutions of second order linear ordinary differential equations whose coefficients are functions of the independent variable.

3.1.2 Existence of Power Series Solutions:

The properties of power series just discussed form the foundation of the power series method. The remaining question is whether a differential equation has power series solutions at all. The answer is simple: if the coefficient P and Q and the function r on the right side of

$$y'' + P(x)y' + Q(x)y = r(x) \tag{3.0}$$

have power series a representation, then (3.0) has power series solutions. The same is true of a, b, c and d in

$$a(x)y'' + b(x)y' + c(x)y = d(x) \tag{3.1}$$

have power series representations and $a(x_0) \neq 0$ (x_0 center of power series). To formulate all of this in a precise and simple way, we use the concept of analytic functions (section 1.6.5 and section 3.1.1 statement 9) and so we state a lemma here.

Lemma 3.3.1: FROBENIUS THEOREM:

If x_0 is a regular singular point of equation, then there exists at least one series solution of the form, where $r = r_1$ is the larger root of the associated indicial equation. Moreover, this series converges for all x such that

$o < x - x_0 < R$, where R is the distance from x_0 to the nearest other singular point (real or complex) of (3.12).

4.0 RESULTS, DISCUSSION AND CONCLUSION:

In this chapter we apply this power series method in finding the general solutions of some of the special ordinary differential equations and discuss their solutions as we intended to do in this work, and to draw a conclusion on the theory of power series method.

4.1 RESULTS

4.1.1 Solution of Airy differential equation

$$y'' - xy = 0 \tag{4.1}$$

The coefficients of the equation (4.1) are everywhere analytic, for convenience we choose our ordinary point $x_0 = 0$

Thus, we assume our solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{4.2}$$

We apply the term wise derivative of (4.2) in (4.1) to obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

Evaluating the second term of the equation above we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Applying shifting of index summation

Let $m = n - 2 \implies n = m + 2$, so the first term becomes

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m$$

And let $m = n + 1 \implies n = m - 1$, the second term becomes

$$\sum_{m=1}^{\infty} a_{m-1}x^m$$

Thus it follows that

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty} a_{m-1}x^m = 0$$

$$2(1)a_2x^0 + \sum_{m=1}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty} a_{m-1}x^m = 0$$

Now the coefficients of

x^0: $\quad 2a_2 = 0 \quad \Rightarrow a_2 = 0 \qquad$ Since $x \neq 0$

x^m: $\quad (m+2)(m+1)a_{m+2} = a_{m-1} = 0$

$$\Rightarrow (m+2)(m+1)a_{m+2} = a_{m-1}$$

$$\therefore\ a_{m+2} = \frac{a_{m-1}}{(m+2)(m+1)}, \qquad m > 0$$

Called the recurrence relation

$$a_3 = \frac{a_0}{3\cdot 2} = \frac{a_0}{6}$$

$$a_4 = \frac{a_1}{4\cdot 3} = \frac{a_1}{12}$$

$$a_5 = \frac{a_2}{5\cdot 4} = 0$$

$$a_6 = \frac{a_3}{6\cdot 5} = \frac{a_3}{6\cdot 5\cdot 3\cdot 2} = \frac{a_0}{180}$$

$$a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_4}{7 \cdot 6 \cdot 4 \cdot 3} = \frac{a_1}{504}$$

Inserting the equation (4.2)

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

We have

$$y = a_0 + a_1 x + \frac{a_0}{6} x^3 + \frac{a_1}{12} x^4 + \frac{a_0}{180} x^6 + \frac{a_1}{504} x^7 + \cdots$$

$$y = a_0 \left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right) + a_1 \left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right) \quad (4.3)$$

Which is the general solution of the Airy differential equation where a_0, a_1 are constants and can be determined by the aid of initial condition.

We apply the power series method to solve Airy differential equation in section 4.1.1 at an ordinary point since the coefficients in the differential equations are everywhere analytic and hence for convenience we choose the ordinary point at $x_0 = 0$.

4.1.4 Lengendre polynomials:

In many applications the parameter n in Legendre's equation will be a non-negative integer. Then the right side of (4.16) is zero when $r = n$ and therefore, $a_{n+2} = 0$, $a_{n+4} = 0$, $a_{n+6} = 0$, ... Hence, if n is even, $y_1(x)$ reduces to a polynomial of degree n. If n is odd the same is true for $y_2(x)$. These polynomials, multiplied by some constants are called Legendre polynomials. Since they are of great practical importance, let us consider them in more detail. For this purpose we solve (4.16) for a_r obtaining

$$a_r = \frac{(r+1)(r+2)}{(r-n)(r+n+1)} a_{r+2} \quad (r \le n-2) \quad (4.21)$$

We may then express all the non-vanishing coefficients in terms of the coefficient a_n of the highest power of x of the polynomial. The coefficient a_n is at first still arbitrary. It is standard to choose $a_n = 1$ when $n = 0$ and

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}, \quad n = 1, 2 \ldots \quad (4.22)$$

The reason is that for this choice of a_n all those polynomials will have the value when $x = 1$; this follows from (4.21) and (4.22) as we then obtain

$$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$$

$$a_{n-2} = \frac{-n(n-1)(2n)!}{2(2n-1)2^n (n!)^2} = \frac{-n(n-1)2n(2n-1)(2n-2)!}{2(2n-1)2^n n(n-1)!\, n(n-1)(n-2)!}$$

$$a_{n-2} = \frac{-(2n-2)!}{2^n (n-1)!\,(n-2)!} \quad (4.23)$$

Similarly,

$$a_{n-4} = \frac{-(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(2n-4)!}{2^n 2!\,(n-2)!\,(n-4)!} \quad (4.24)$$

And so on,

In general, when $n - 2m \ge 0$ we obtain

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!} \tag{4.25}$$

Hence, the resulting solution of Legendre's differential equation (4.9) is called the Legendre polynomial of degree n and is denoted by $P_n(x)$.

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-m)!}{2^n m!\,(n-m)!\,(n-2m)!} x^{n-2m} \tag{4.26}$$

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!\,(n-1)!\,(n-2)!} x^{n-2} \pm \cdots \tag{4.27}$$

The first few of these functions are

$$P_0(x) = 1,$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1),$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),$$
$$P_1(x) = x,$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x),$$
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

4.1.5 Solution of Bessel Differential Equation

$$x^2 y'' + xy' + (x^2 - v^2)y = 0 \tag{4.28}$$

Where the parameter v is a non-negative number.

Normalizing the equation we have

$$y'' + \frac{1}{x} y' + \left(1 - \frac{v^2}{x^2}\right) y = 0 \tag{4.29}$$

This is a differential equation of the form

$$y'' + b(x)y' + c(x)y = 0 \tag{4.30}$$

Comparing (4.29) and (4.30) we obtain

$$b(x) = \frac{1}{x}, \qquad c(x) = 1 - \frac{v^2}{x^2}$$

Clearly $x = x_0 = 0$ is a singular point that can be removed, since

$$xb(x) = 1, \qquad x^2 c(x) = x^2 - v^2$$
$$b_0 = \lim_{x \to x_0 = 0} xb(x) = 1$$

$$c_0 = \lim_{x \to x_0 = 0} x^2 c(x) = \lim_{x \to x_0 = 0} x^2 - v^2 = -v^2$$

Since $x = 0$ is a regular singular point we seek a solution to the Bessel equation of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad a_0 \neq 0 \tag{4.31}$$

Where r must satisfied the indicial equation given by

$$r(r-1) + b_0 r + c_0 = 0 \tag{4.32}$$
$$r(r-1) + r - v^2 = 0$$
$$(r-v)(r+v) = 0$$
$$=> \quad r = \pm v$$

Now, using $r = v$, we have that

$$y = \sum_{n=0}^{\infty} a_n x^{n+v} \tag{4.33}$$

$$y' = \sum_{n=0}^{\infty} (n+v) a_n x^{n+v-1} \tag{4.34}$$

$$y'' = \sum_{n=0}^{\infty} (n+v)(n+v-1) a_n x^{n+v-2} \tag{4.35}$$

Putting in (4.28) we have

$$x^2 \sum_{n=0}^{\infty} (n+v)(n+v-1) a_n x^{n+v-2} + x \sum_{n=0}^{\infty} (n+v) a_n x^{n+v-1} + (x^2 - v^2) \sum_{n=0}^{\infty} a_n x^{n+v} = 0$$

Implies that

$$\sum_{n=0}^{\infty} (n+v)(n+v-1) a_n x^{n+v} + \sum_{n=0}^{\infty} (n+v) a_n x^{n+v} + \sum_{n=0}^{\infty} a_n x^{n+v+2} - \sum_{n=0}^{\infty} v^2 a_n x^{n+v} = 0$$

Applying shifting of index summation

Let $\mathrm{m} = \mathrm{n} + 2 \;\; => \;\; \mathrm{n} = \mathrm{m} - 2$ then the third term becomes

$$\sum_{n=0}^{\infty} a_n x^{n+v+2} = \sum_{m=2}^{\infty} a_{m-2} x^{m+v} = \sum_{n=2}^{\infty} a_{n-2} x^{n+v}$$

Thus, it follows that

$$\sum_{n=0}^{\infty} (n+v)(n+v-1) a_n x^{n+v} + \sum_{n=0}^{\infty} (n+v) a_n x^{n+v} + \sum_{n=2}^{\infty} a_{n-2} x^{n+v} - \sum_{n=0}^{\infty} v^2 a_n x^{n+v} = 0$$

$$v(v-1) a_0 x^v + (v+1) v a_1 x^{v+1} + \sum_{n=2}^{\infty} (n+v)(n+v-1) a_n x^{n+v} + v a_0 x^v + (v+1) a_1 x^{v+1}$$

$$+ \sum_{n=2}^{\infty} (n+v) a_n x^{n+v} + \sum_{n=2}^{\infty} a_{n-2} x^{n+v} - v^2 a_0 x^v - v^2 a_1 x^{v+1} - \sum_{n=0}^{\infty} v^2 a_n x^{n+v} = 0$$

$$a_0[v(v-1) + v - v^2] x^v + a_1[(v+1)v + (v+1) - v^2] x^{v+1}$$

$$+ \sum_{n=2}^{\infty} \{a_n[(n+v)(n+v-1) + (n+v) - v^2] \; + \; a_{n-2}\} x^{v+n} = 0 \tag{4.36}$$

$$r = v, \quad a_0 \neq 0, \quad a_1 = 0, \quad 2v + 1 = 0 \;\; => \;\; v = -1/2$$

$$a_n[n(n+2v)] + a_{n-2} = 0 \tag{4.37}$$

Or

$$a_n = \frac{-a_{n-2}}{n(n+2v)}, \qquad n = 2, 3, \ldots \tag{4.38}$$

Called recurrence formula and

$0 = a_1 = a_3 = a_5 = \cdots$.

Thus we obtain

$$a_2 = \frac{-a_0}{2 \cdot 2(1+v)},$$

$$a_4 = \frac{-a_2}{4 \cdot 2(2+v)} = \frac{a_0}{4 \cdot 2^3 (1+v)(2+v)},$$

$$a_6 = \frac{-a_0}{2^6.\,3!\,(1+v)(2+v)(v+3)}$$

Hence, in general we have

$$a_r = \frac{(-1)^{r/2} a_0}{2^r.\,(r/2)\,!\,(1+v)(2+v)(v+3)\dots(v+r/2)}\,, \quad \text{for even } r \tag{4.39}$$

The resulting series solution is therefore

$$y_1 = a_0 x^v \left\{1 - \frac{x^2}{2^2(v+1)} + \frac{x^4}{2^4\cdot 2!\,(v+1)(v+2)} - \frac{x^6}{2^6\cdot 3!\,(v+1)(v+2)(v+3)} + \cdots\right\}$$

This is valid provided v is not a negative integer.

Similarly, when $r = -v$, we obtain

$$y_2 = b_0 x^{-v} \left\{1 + \frac{x^2}{2^2(v-1)} + \frac{x^4}{2^4\cdot 2!\,(v-1)(v-2)} + \frac{x^6}{2^6\cdot 3!\,(v-1)(v-2)(v-3)} + \cdots\right\}$$

This is valid provided v is not a positive integer.

Therefore except for these two restrictions, the complete solution of Bessel's equation is

$$y = y_1 + y_2 \tag{4.40}$$

with two arbitrary constants a_0 and b_0.

4.1.6 Bessel Functions:

It is convenient to present the two results above in terms of gamma functions, remembering from section 1.6 (chapter one) that for $x > o$

$$\Gamma(x+1) = x\Gamma(x)$$
$$\Gamma(x+2) = (x+1)\Gamma(x+1) = (x+1)x\Gamma(x)$$
$$\Gamma(x+3) = (x+2)\Gamma(x+2) = (x+2)(x+1)x\Gamma(x)$$

and so on.

If at the same time, we assign to the arbitrary a_0 in (4.39) the value $\frac{1}{2^v\Gamma(v+1)}$, then we have for $r = v$,

$$a_2 = \frac{-1}{2^2(1+v)} \cdot \frac{1}{2^v\Gamma(v+1)} = \frac{-1}{2^{v+2}(1!)\Gamma(v+2)}$$

$$a_4 = \frac{-1}{4\cdot 2(2+v)} \cdot \frac{-1}{2^{v+2}(1!)\Gamma(v+2)} = \frac{1}{2^{v+4}(2!)\Gamma(v+3)}$$

And

$$a_6 = \frac{-1}{2^{v+6}(3!)\Gamma(v+4)}$$

We can see the pattern taken shape

$$a_r = \frac{-1}{2^{v+r}\left(\frac{r}{2}!\right)\Gamma\left(v+\frac{r}{2}+1\right)}\,, \quad \text{for even r} \tag{4.41}$$

Therefore putting $r = 2k$, (4.41) becomes

$$a_{2k} = \frac{(-1)^k}{2^{v+2k}(k!)\Gamma(v+k+1)}, \qquad k = 1, \quad 2, \quad 3\dots$$

Therefore, we write the new the form of the series for y_1 as

$$y_1 = x^v\left\{\frac{1}{2^v\Gamma(v+1)} - \frac{x^2}{2^{v+2}(1!)\Gamma(v+2)} + \frac{x^4}{2^{v+4}(2!)\Gamma(v+3)} - \cdots\right\} \tag{4.42}$$

This is called the Bessel function of the first kind of order v and is denoted by $J_v(x)$.

$$J_v(x) = \left(\frac{x}{2}\right)^v \left\{\frac{1}{\Gamma(v+1)} - \frac{x^2}{2^2(1!)\Gamma(v+2)} + \frac{x^4}{2^4(2!)\Gamma(v+3)} - \cdots\right\} \tag{4.43}$$

This is valid provided v is not a negative integer.
If we take the other value $r = -v$, the corresponding result becomes

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v}\left\{\frac{1}{\Gamma(1-v)} - \frac{x^2}{2(1!)\Gamma(2-v)} + \frac{x^4}{2^2(2!)\Gamma(3-v)} - \cdots\right\} \tag{4.43}$$

Provided that v is not a positive integer.
In general terms

$$J_v(x) = \left(\frac{x}{2}\right)^{v}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(v+k+1)} \tag{4.44}$$

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{v}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(v-k+1)} \tag{4.45}$$

$J_v(x)$ and $J_{-v}(x)$ are two linear independent solutions of the original equation (4.28). Hence, the complete solution is

$$y = AJ_v(x) + B\,J_{-v}(x) \tag{4.46}$$

Where A and B are constants.
Some Bessel functions are commonly used and worthy of special mention. This arises when v is a positive integer, denoted by n;

$$J_n(x) = \left(\frac{x}{2}\right)^{n}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(n+k+1)} \tag{4.47}$$

Where

$$\Gamma(k+1) = k!\ , \qquad k = 0, \quad 1, \quad 2 \ldots$$
$$\Gamma(n+k+1) = (n+k)!$$

And the result of (4.47) then becomes

$$J_n(x) = \left(\frac{x}{2}\right)^{n}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)(n+k)!} \tag{4.48}$$

On the whole, we have seen that $J_v(x)$ and $J_{-v}(x)$ are two solutions of the Bessel's equation. When v and $-v$ are not integers, the two solutions are independent of each other. Then $y = AJ_v(x) + B\,J_{-v}(x)$ When however, $v = n$ (integer), the $J_n(x)$ and $J_{-n}(x)$ are not independent, but are related by $J_{-n}(x) = (-1)^n J_n(x)$.
This can be shown by referring again to our knowledge of gamma functions.

$$\Gamma(x+1) = x\Gamma(x)\ , \qquad => \Gamma(x) = \frac{\Gamma(x+1)}{x}$$

and for $(x \leq 0)$ negative integral values of x, or zero, $\Gamma(x)$ is infinite from the previous result.

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(v-k+1)}, \quad k = 0, 1, 2 \tag{4.49}$$

Let us consider the gamma function $\Gamma(v-k+1)$ in the denominator of (4.49) and let v approach closely to a positive integer n. Then, $\Gamma(v-k+1) \rightarrow \Gamma(k-n+1)$ when $k-n+1 \leq 0$ $i.e.\, k \leq (n-1)$, then $\Gamma(k-n+1)$ is infinite. The first finite value of $\Gamma(k-n+1)$ occurs for $k = n$. When values of $\Gamma(k-v+1)$ are infinite the coefficients of $J_{-v}(x)$ are zero.
The series, therefore, starts at $k = n$

$$\therefore\ J_{-n}(x) = \left(\frac{x}{2}\right)^{-n}\sum_{k=n}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(k-n+1)}$$

$$J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k x^{2k-n}}{2^{2k-n}(k!)\Gamma(k-n+1)}$$

Putting $k = p + n$ we obtain

$$J_{-n}(x) = \sum_{p=0}^{\infty} \frac{(-1)^{p+n} x^{2p+n}}{2^{2p+n}(k!)(k-n)!}$$

$$= (-1)^n \sum_{p=0}^{\infty} \frac{(-1)^p x^{2p+n}}{2^{2p+n}(p!)(p+n)!}$$

$$= (-1)^n \left(\frac{x}{2}\right)^n \sum_{p=0}^{\infty} \frac{(-1)^p x^{2p}}{2^{2p}(p!)(p+n)!}$$

$$= (-1)^n \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)(k+n)!}$$

$$\therefore J_{-n}(x) = (-1)^n J_n(x)$$

So after all that, the series for $J_n(x)$ is

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{\frac{1}{\mathrm{n}!} - \frac{1}{(1!)(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!}\left(\frac{x}{2}\right)^4 - \cdots\right\} \qquad (4.50)$$

From this we obtain two most commonly used functions

$$J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots \qquad (4.51)$$

And

$$J_1(x) = \frac{x}{2}\left\{1 - \frac{1}{(1)!\,(2!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(3!)}\left(\frac{x}{2}\right)^4 - \cdots\right\} \qquad (4.52)$$

4.3 Conclusion:

Ordinary differential equations have been of a great relevance to the world scientists and serve as techniques to model many physical problems.

In this research work, we studied explicitly power series and observed that, power series is of great use in solving all ordinary differential equations and more importantly to those ordinary differential equations with variable coefficients; which are cumbersome and complex and cannot be solved by elementary methods analytically, we proceeded to define some special ordinary differential equations and obtained their solutions through the power series solution method.

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