Teaching and Learning of Differential Equation: A Critical Review to Explore Potential Area for Reform Movement

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Abstract: *The role of differential equations (DEs) is very important in the modern technologies to inter-relate and solve a variety of routine or daily life problems. Several approaches have been developed and more are being developed to make DEs course more effective and valuable. However, a limited work is available that may enable the comparison of these methods. Therefore, in this study, articles for a 19-year period, from 1996 to 2015, have been critically reviewed and analyzed. Five main categories including, algebraic, graphical, numerical, technological and inquiry oriented based approaches have been identified and compared. Algebraic based approaches are based on the several steps to solve the DEs analytically, while graphical methods are the qualitative in nature, used to analyze either a graph or direction fields to solve a DE problem. Numerical methods offer the solution of DEs through appropriate approximations. Recently development in the technology has integrated these three categories into single approach. Inquiry oriented based approaches have further added the positive effect of environment, epistemological and motivational beliefs on the DEs learning. This study will open new avenues related to DEs learning and will enable to indicate the potential area in which the further research is required.*

Key Words: *Algebraic methods, differential equations, epistemological and motivational beliefs, qualitative methods, technology based approaches*

1. INTRODUCTION:

Differential equations (DEs) have an essential role in mathematics and have been at the center of calculus for centuries. The study of differential equations started in the late 17th century with Sir Isaac Newton, who sought information about motion of planets indirectly through the analysis of rate of change equations (Rasmussen et al., 2000). The concept of DE is used to model and understand real life problems. Therefore, these provide opportunities to formulate the application of phenomena from other discipline of science and social science fields such as Physics, Astronomy, Biology, and Economics.

Teaching and learning of differential equation is the most difficult part of the mathematics course, particularly at pre-university level. This is because, the topic of differential equation along with differentiation and integration is introduced first time at 12th year of study, and the students have no previous knowledge and understandings of this topic (Rehman et al., 2012). In addition to it, students' special attention, efforts and learning strategies are required to solve problems containing differential equations, particularly non-routine problems because these problems are typically concerned with unanticipated, unusual, and strange solutions (Polya, 1981; Rehman et al., 2012).Even talented calculus problem solver sometimes became unable to solve these non-routine problems(Dawkins et al., 2014). So it is common for students to avoid the essential part of mathematics, which leads to sever understanding problems at higher levels of education, when they correlate the real life problems.

From the teaching point of view, most important aspects of the recent studies in mathematics education were to find competent strategies for teaching differential equation course. Within the case of differential equation, various kinds of scheme have been come out to deal with the concepts related to them(Raychaudhuri, 2008). Generally, three different approaches (algebraic, numerical and graphical) are employed to solve differential equations (Arslan, 2010b; Artigue, 1989). In traditional differential equation teaching and learning, algebraic approach predominates, while recently, graphical and numerical approaches are more emphasized to facilitate the conceptual learning of differential equations.

In a traditional differential equation environment, Selahattin (2010b) reported that nature of students' learning is procedural and is restricted to mastering and applying a few algebraic techniques. Artigue (1989) supported these results in a sense, so as to learner do not have understanding of differential equation concept (Boyce, 1994; Rasmussen, 2001). The main difficulties that students found when handling the algebraic based solutions of differential equations are related to the unsuitable choice of the method of solution or an incorrect process of integration (Camacho-Machín et al., 2012b).

Graphical based solutions are considered as qualitative approach and show the real understandings of the students. However, in graphical based solutions, different functions such as linear, exponential, and trigonometric and hyperbolic functions are difficult to represent and retrieve (Camacho-Machín et al., 2012a). In addition to these, transition from the algebraic to the graphical register is quite hard and students often make mistakes during this conversion (Camacho-Machín et al., 2012a).Geometric representation can be incorporated to provide sense to the solution methods for ordinary differential equation while modeling different phenomena (Camacho-Machín et al., 2015; Rowland et al., 2004). However, it has proven to be critical for the learners to adapt a geometrical approach for solving differential equations and their solutions.

In mid 1980s, the reform movement regarding to teaching of differential equations was stimulated by increased accessibility of technology and by calculus reform. The use of technological advances as a reform movement in teaching and learning differential equations has been initiated to analyze ordinary differential equations regarding to algebraic, numerical and graphical representations (Camacho-Machín et al., 2015; Hubbard et al., 2012). This moment yielded better results due to awareness of technological advances at higher levels of education (Rowland, 2006; Rowland et al., 2004).Further this movement moved a step ahead by means of introducing the inquiry-oriented approach to the teaching and learning of differential equations (Gado, 2005; Ju et al., 2007).

Literature reveals that various researchers have done their best to identify the difficulties arose during teaching and learning differential equation course and they also tried to overcome these issues by employing different approaches. However, up to researchers' knowledge, no one has critically reviewed and compared these studies in the last decade. Therefore, in this study, articles regarding to the issues and of teaching and learning differential equations problem solving and different approaches have been reviewed.

2. RESEARCH OBJECTIVES:

The aim of current article is to review the research approaches or movements regarding to the issues of teaching and learning differential equations problem solving. It is anticipated that the findings of current study will assist students, educators and researchers with some insightful ideas about the pattern or approaches, and issues studied in the area of differential equations.

The research questions addressed by this study are therefore:

1. What are the main approaches or classifications for teaching and learning of differential equation problem solving?
2. What are the recent trends in the DEs problem solving between the years 1996 to the year 2015 from these hundred selected journal articles.
3. How technology advances and inquiry oriented differential equation projects contribute towards students understanding of differential equation concepts?
4. What is the potential area, particularly at pre-university level, in which the further research is required?

The following sections have demonstrated the introductory approaches, and tabulated form of the reviewed articles. The last section summaries our conclusions and some implications of results and unresolved issues for the future research.

3. METHODOLOGY:

Literature has been keenly investigated which showed that several researches and research projects have been conducted on teaching and learning differential equation different. Initially, teaching DEs were classified as, traditional DEs teaching and contemporary DEs teaching. However, integration of technological advancements and addition of recent researches, teaching and learning DE course may be classified into 5 main groups such as, analytical, graphical, numerical, reform movements. The reform movements further include incorporation of technology and Inquiry-oriented differential equations (IO-DE) projects, psychological factors. In addition to these, several researchers also worked on other perspectives, such as concept maps and vee diagrams etc.

Major Techniques Related to Teaching and Learning of DE

Figure 1 illustrates the frame work for the current work. Detail of each method is provided in the following sections.

Algebraic /Analytic Techniques

Algebraic methods are mostly procedural in nature, involving different symbols, steps and techniques to solve differential equations problems. Traditionally, to solve differential equation means to find a value for unknown, consequently it is essential to find an expression for the unknown function (Habre, 2003). Analytically, a solution to a differential equation is a function that satisfies the ordinary differential equation. Generally, ODE equations are categorized as linear, separable, exact, and others.

An analytic method of solution is offered the student for each class of equation whereas integration is essential in the solution process. Hence integration, formulas and steps are important in these kinds of approaches. Both of the general and exact solutions are possible through these approaches. At higher university levels, partial differential equations are solved using these techniques. Everywhere the answer connected to the algebraic solution of DEs are concerned, errors stemmed from student's misunderstanding over equation type with another or symbolic errors (Arslan, 2010b).

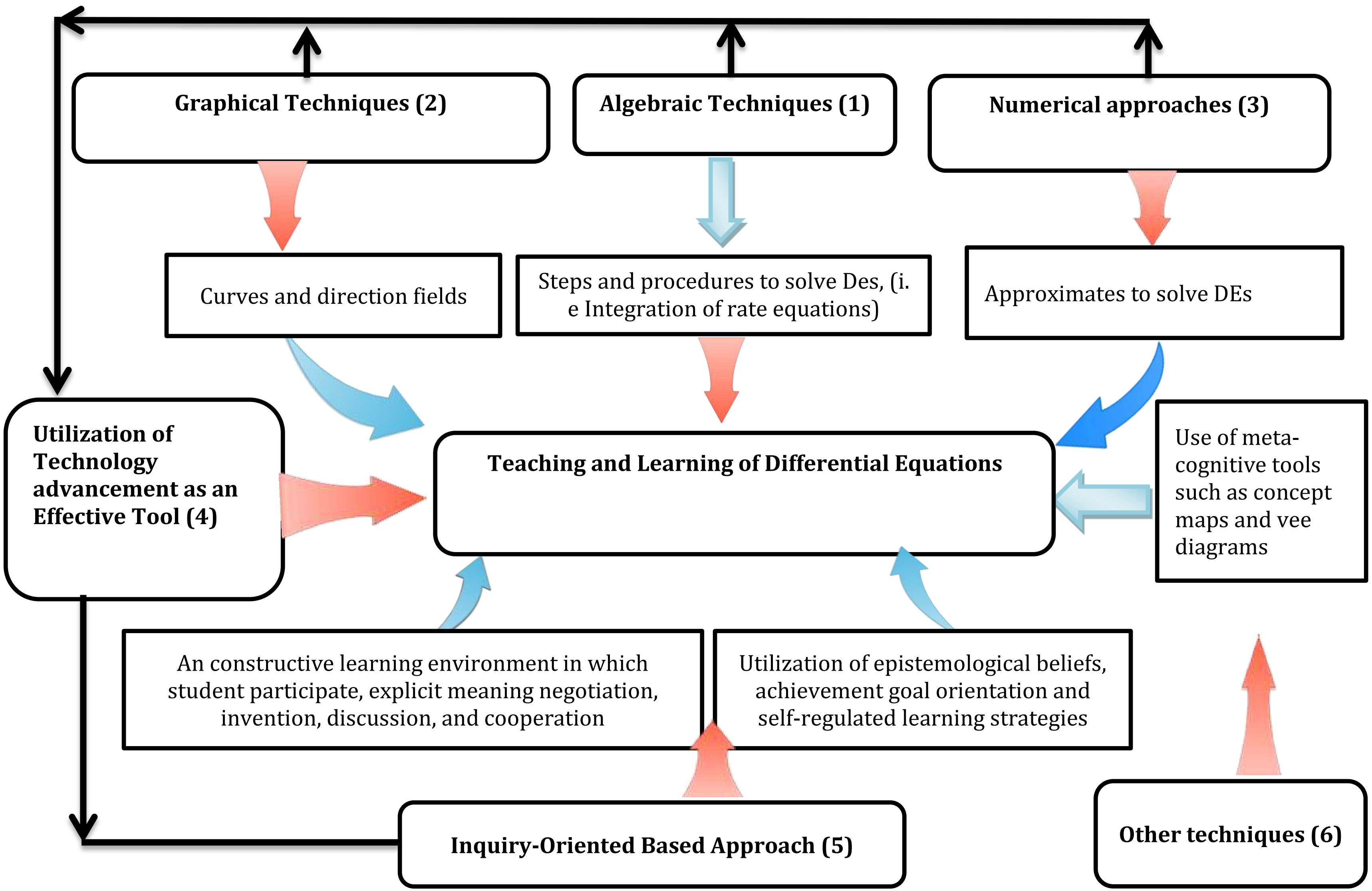

Figure 1: Framework for Literature Review

With less training and efforts, teachers can be effectively taught the course by applying few procedures and steps. According to Habre (2000), mostly students prefer algabraic method, arguing that this method gives exact answer.However, beside of their simplicity, several drawbacks limit their use. Although in a traditional differential class, nature of students' learning is procedural, but is restricted to mastering and applying some algebraic techniques (Arslan, 2010b). The main difficulties that students found when handling the algebraic based solutions of differential equations are related to the unsuitable choice of the method of solution or an incorrect process of integration (Camacho-Machín et al., 2012b). It is align with the Artigue (1989) in a sense, that students have misunderstanding and learning difficulties relating to differential equations (Boyce, 1994; Rasmussen, 2001).

In addition, regarding to the interpretation of algebraic expression, Rowland et al. (2004) documented some difficulties while students interpreted the terminologies involved in an ordinary differential equation (ODE) contextually, and whilst they converted the context into mathematical form. In their study, only one student from physics background were able to recognize dv/dt as representing acceleration and recognized that they have to find out an equation for acceleration instead of velocity. Rowland (2006) reported that students thought in term of rate of change. However, difficulties also take place when they conceptualized a constant as the rate of change as that term 'constant' is somewhat explicitly static, hence it could not be a rate of change. Similar results are reported by Arslan (2010b). The cause for the students' failure to comprehend the issue may be the course content and instruction of differential equations course (Blanchard, 1994; Boyce, 1994). Several researchers suggested that traditional instructions are not adequate particularly for conceptual learning and hence contemporary approaches in teaching and learning are required (Arslan, 2010a; Boyce, 1999; Habre, 2003; Rasmussen et al., 2000).

Therefore, Mallet et al. (2009) illustrated constructive, discovery-based approach in teaching and learning, wherein students utilize their accessible skills as a frame work for the accumulation of ordinary differential equations solution. This strategy is online with the 'Five E's' of constructivism: engage, explore, explain, elaborate and evaluate(Bybee, 2001). Moreover, instead of relying on traditional instructional methods, Mallet et al. (2009) replaced the teaching resources, includes worksheets, lecture notes, and demonstrating materials to teaching resources as align with constructive approach.

Zandieh et al. (1999) and Habre (2003) have differentiated the students conceptual understanding of the meaning of an ODE and its solution. In these studies, open ended question were posed about differential equation concept and its solution concept and students given the correct definitions, although algebraically. Habre (2003) employed the algebraic problem solver. However, the target students were failed to conceptualize DEs and their solution. These students were not succeeded to classify the type of differential equation. Few have also used incorrect symbols while solving ordinary differential equation algebraically. Similarly, Camacho-Machín et al. (2012a) documented that for the verification of the solution function, students tried to use algorithm strategy (substitution method or direct method), instead of relying on concepts associated with it. Earlier few researchers also reported that ideas from basic concepts of pre-calculus and calculus such as mathematical differences among quality, rate and rate of change (Rowland, 2006; Rowland et al., 2004), equation and solution (Raychaudhuri, 2008) are essential for the achievement of DEs course. In addition, Czocher et al. (2013) illustrated that vital concept such as functions, derivatives, integrals, are vastly epistemological mismatches between calculus and differential equation. However, there is a strong interrelation with these mathematical concepts and differential equation. For example, the concept of derivative might be the bridge for the students as it can relate the derivative concept with the prime concepts that appeared in an ordinary differential equation. Czocher et al. (2013) also evidently proved that differential equation concepts needs capability in calculus concepts and skills.

Nevertheless, students possess the conceptual resources such as differentiation, integration, function and graphical representation of function derivative, but they are unable to utilize these resources (Camacho-Machín et al., 2012b). The study of Hu et al. (2013) is align with it, in a sense that its focal point was on how students make use of mathematical resources for instance differentiation and integration in physics problem solving and has contribution to both fields mathematics and physics.

4. GRAPHICAL / QUALITATIVE TECHNIQUES

Graphical based solutions are considered as qualitative approach and show the real understandings of the students. Most of the time various analytic methods are not enough to solve differential equations; however, numerical and qualitative methods are applicable towards all differential equations problems. Therefore, several authors (Camacho-Machín et al., 2008; Machín et al., 2009) suggested that accepting the system of graphical representation can widen the understanding of differential equation solution concept. However, Habre (2000) argued that analytic approach is not obsolete, in many instances it goes along with qualitative approach both hand-in-hand.

Graphical representation played a major role in the study of Rasmussen (1996), where student used graph of differential equation and its solution curve and direction fields, to qualitatively analyzed a differential equation solution. Graphically or geometrically, a graph of solution is a function that has the characteristics such as increasing, decreasing, rate of increasing decreasing and these are prescribed by the differential equation. Hence, geometric representation might be able to incorporate to give sense to solution methods for ordinary differential equation while modeling different phenomena (Camacho-Machín et al., 2015; Rowland et al., 2004).

However, graphical based solutions, different functions such as linear, exponential, and trigonometric and hyperbolic functions are difficult to represent and retrieve (Camacho-Machín et al., 2012a). Besides this, Camacho-Machín et al. (2008) also documented that students who used algebraic method to solve DEs but these students were failed in graphical representation and mathematical interpretation. Beside these, transition from the algebraic to the graphical register is quite hard and students often make mistakes during this conversion (Camacho-Machín et al., 2012a).

Numerical Approaches

Likewise qualitative methods, numerical methods offer alternate solutions for DE which cannot be solved using analytic methods(Rasmussen, 1998). Numerical methods give access to approximate solutions of differential equations, while qualitative or graphical methods views the differential equations geometrically as well as analyzing the differential equations itself, hence these methods provide overall information about solutions (Rasmussen, 1998). Several researchers described and compared that how numerical methods are used to analyze differential equations(Blanchard et al., 2002; Borrelli et al., 1996; Coombes et al., 1995; Kostelich et al., 1996; Lomen et al., 1996; West, 1996).

Further, (Rasmussen, 1998) investigated students' understanding of and difficulties while utilizing qualitative and numerical methods to analyze differential equations. Findings showed that difficulties influencing students understanding includes function dilemma, trend of overgeneralization such as, overgeneralization of the autonomous term, and interference from informal concepts, and the complications with graphical interpretations. Several researches extended this concept and also stated that in order to support the development of understandings, the rule of three algebraic, numerical, and graphical should be replace with rule of four, where writing in mathematics has an important role.

Writing Techniques

In addition to algebraic, graphical and numerical methods, several researchers also have focused the writing techniques as the effective learning tool for differential equation learning and solving (Meier et al., 1998; Porter et al., 1995). Habre (2002) reported that students considered writing an important tool that can be used to improve differential equations learning as well as elucidate ideas geometrically presented.

Utilization of Technology advancement as an Effective Tool for DE

The reform movement within the region of differential equations was stimulated in the mid-1980s by the improved accessibility of technology and by calculus reforms for the effective teaching and learning differential equations (Ju et al., 2007). Use of technological advances was initiated to analyze ordinary differential equations by combining algebraic, numeric, and graphical representations (Camacho-Machín et al., 2015; Hubbard et al., 2012). Examples of technology incorporations are Mathlets; a java applet, Computer Algebra System including Maxima, Maple, and Math lab and Interactive Differential equation. These software pogrammes are employed to visualize graph and also to understand the connection between graph and equations (Azman et al., 2013; Dana-Picard et al., 2008). These developments have changed the entire setting and opened a new evenues for assembling the concepts and their connections to the real world situations (Devaney et al., 1998). As a result, variety of real world problems, including moving object velocities/accelrations, temperature/pressures changes, fluid flows and aerodymomics were identified and mechanisms were deveolped to solve these problems though solving their realted differential equations (Aravind et al., 2013; Moore et al., 2013; Pollak, 2015; Yurtseven Avci et al., 2014).

Inspite of the many advanatge of technology advanced tools, modeling a physical problem through differential equation became a problematic situation in reality. Because, students emerging from a conventional differential equation courses have slight understanding of whatsolutions of differential equation represent in an applicable conditions (Habre, 2000). Hence, several educationalist considered the importance of qualitative approach and it should be adopted for differential equation course. Although in past, adoptation of such approach was not accepted due to the difficulties related to visual aspects. However, incorporation of computer graphics has provided extraordinary visual capabilities to the teachers and learners ang give advantage in the visualization of complex relationship that student often found difficult to understand. Visualization has a very important role regarding to the understanding of dynamical aspects of an basics differential equations. It is used to assists in understanding the derivative as the slope of a curve; and also help to interprate graphs,and reading information from these graph such as existence of an equilibrium state and long-term behaviour of the solutions (Borrelli et al., 1999). In this respect, Gollwitzer (1991) considered the direction field as an important tool that is used to encourage students to think about visual component in differential equation.

Two main processes that are more worthy to mention in the study of ODE are; identification of equilibrium solution and the recognition of the values where the slope field (Camacho-Machín et al., 2015). Habre (2000) analysed how students use the slope field to solve first order differential equation and how they extract information from these field in a reform setting. Findings were highly encourging. Howerver, student work with ease with single representation and they found it difficult to cope with different representations simultaniously (Habre, 2000). Rowland et al. (2004) explored that teaching and learning become more problemetic to handle different register of representation in the context of ODE. This problem is associated with students understanding of how mathematical model are interelated with the real context and how they interpret the parameters. Camacho-Machín et al. (2015) revealed that different digital tools gave students self-confidence to represent the same phenomena and also encouraged them to empower the information using graphical and as well as numerical representation of solution. Furthermore, by considering the relationship between ODE and context, and with the addition of digital tool can assist studentsin their understanding, in many countries,differential equations curriculumhas been changed at introductory level (Rasmussen, 1996).

Despite of technology advantages, at initial or pre university levels, it is a great challenge to determine students interaction by means of the digital tools and representation guidance related with ordinary differential equations to provide sense to parameters connected with it (Rowland, 2006; Rowland et al., 2004). Also, it is difficult task to find out how to construct instruction strategies to promote student learning (Rasmussen, 2001). Particularly the developing countries, utilization of the technological based methods are still challenging.

Inquiry-Oriented Based Approach for learning Differential Equations

The inquiry-oriented class for differential equation learning is an constructive learning environment where student participate, explicit meaning negotiation, create, discuss, and cooperate by integrating students' mathematical understandings to attain the formal mathematics (Gado, 2005; Rasmussen et al., 2000). Therefore, students inquiry enables students to learn mathematics through engagement in authentic reasoning and also make them as a authorize learner to glance mathematics as a human activity as well as they are capable to reinvent mathematics (Rasmussen et al., 2007b).

This approach was established on the recommendations of international commission on teaching and learning of mathematics at university level to overcome the newest challenges. Among these, one major issue is the accommodation of much large and diverse group of students (Holton, 2001).

Through inquiry-oriented differential equations (IO-DE) approach students are able to learn most up-to-date mathematics via inquiry, which engrossed in engaging in mathematical conversation, creating and subsequently conjecturing, exposing and defending ideas and their approach to solve innovative problems. In contrast to it, in traditional environment, instructions design discourages individuals as of generating their own problem solving strategies.

Beside this, several researches have also given the concept of realistic mathematics education approach (RME) which is an alternative approach to the traditional curriculum approach, in which students' learning is based on practical real condition (Rasmussen et al., 2000; Yackel et al., 2000). Therefore, to adapt instructional design theory of realistic mathematics education approach (RME) is key stone of inquiry-oriented differential equation project (Rasmussen et al., 2007a), in which students' find out solution methods and make interaction with teacher as well as with class mates (Kwon, 2002).

The study of Rasmussen et al. (2007a) has a significant contribution to the RME emergent model, in which the researchers elaborated the emergent model for student reinvention of system of linear differential equations solutions. In addition to it,Rasmussen et al. (2006b) also elucidated the function of graphs and gestures for the reinvention of the Euler method for differential equation and emphasized how these functions change in students' subsequence use of the Euler method to approximate system of differential equations. This study provided a dictionary of student gestures and also how they are associated with students' reinvention and use of algorithm of Euler method.

A second corner stone of Inquiry-oriented differential equations (IO-DE) project is a research area whose focal point is student thinking and teacher knowledge (Rasmussen et al., 2007b). Regarding to student thinking, Keene et al. (2011) observed students' ideas about the use of time as dynamic quantity and the way time-based reasoning can promote understanding of solution function. Author identified five different technique where students had integrated time as a varying quantity as their understanding of differential equation increased.

Beyond mathematics content knowledge, Wagner et al. (2007) identified different forms of knowledge that are very important o IO-DE teaching. They have studied a case of a mathematician who employed this reform for the first time. Authors argued that reform practice of instructions include knowledge apart from mathematics pedagogical knowledge, content knowledge and pedagogical content knowledge supporting traditional instructions design.

Literature from qualitative studies has also evidently proved that inquiry oriented differential equation approach enhances desirable students learning outcomes as compared to traditional assessment methods (Rasmussen et al., 2007b). Regarding quantitative assessments of IO-DE learning, Rasmussen et al. (2006a) evidently proved that there was no significant difference on student performance on the procedurally-oriented items, although analytic solutions were the main focus of the comparison groups. However, IO-DE group students performed better on the conceptually-oriented than the TRA-DE (traditional differential equation students) group. In similar context, Kwon et al. (2005) conducted a follow-up study one year after instruction for a subset of the students from the comparison study on the retention effects of conceptual and procedural knowledge. Researchers concluded that IO-DE enables students to emphasize on variety of strategies both simultaneously and with equal importance, due to which, these students retained multiple ways to approach problems and performed better even after one year (Kwon et al., 2005). Other evaluation studies also proved the optimistic outcome of IO-DE approach on students' conceptual understanding, problem solving, retention and justification (Ju et al., 2004; Kim et al., 2006; Kwon et al., 2005; Rasmussen et al., 2006a).

In addition to mathematics conceptions, Cobb (1985) argued for the assessment of students' belief systems apart from these reforms. Ju et al. (2007) illustrated the effect of an inquiry oriented differential equation course on the enhancement of student beliefs about mathematics. Authors extended the evaluation of the inquiry oriented differential course model beyond the cognitive aspects of mathematics learning and investigated transformation of students' beliefs about mathematics as well as their relation to the discipline. Concerning the students' mathematical growth, several researchers also highlighted the significance of social or cultural processes (Cobb, 1995; Lave, 1988; Rogoff, 1990; Saxe, 1991). An another remarkable approach was carried by Yackel et al. (2000). They extended the analysis of social interaction patterns, social and socio mathematical norms regarding explanation and how these norms were characterized in differential equation class.

Several researches also claimed that student learning is located inside the interconnected constructs of problem solving, epistemology, and self-regulated learning (SRL) (Muis, 2007; Muis et al., 2009; Schommer-Aikins, 2004; Stockton, 2010). Typically, successful problem-solvers exert control over the problem space and have availing epistemological beliefs (Muis, 2008; Perels et al., 2005; Schoenfeld, 1983, 1985, 1989). Schommer-Aikins (2004) hypothesized reciprocal relationship between epistemological beliefs and self-regulated learning. However, experimental results have shown that a relationship exists between SRL and epistemological beliefs in multiple contexts (Bråten et al., 2005; Hofer, 1999; Muis, 2008).The several studies also highlighted thefunction of self-efficacy beliefs, motivation, engagement, and attitudes towards mathematics learning (Abdulwahed et al., 2012; Alpaslan et al., 2016; Fadlelmula et al., 2015; Velayutham et al., 2012).Beside these, role of goal orientation beliefs (part of self-motivational beliefs) were also found as an energizing agent for an individual's self-regulatory behaviors and influence the implementation of self-regulatory knowledge and skills (Kingir et al., 2013; Montalvo et al., 2004). Wolters et al. (1996) studied the relationship between three goal orientations and student self-regulated learning focusing the subject mathematics.

So, for the reformed movement, epistemological beliefs about differential equation problem solving, self-regulated learning (SRL) and achievement goal orientation may play an important role towards the students' conceptual understandings in teaching and learning of differential equation.

5. OTHERS PERSPECTIVES:

In addition to aforementioned five main categories, related to teaching and learning of differential equation course, several researchers also focused on the alternative ways. In this context, researchers proposed the addition of more qualitative or conceptual type questions into the curriculum and also discovery of errors in their reasoning techniques. The analysis of error is another powerful tool that may provide important information regarding to students' mistakes while working in any mathematical domain. Sánchez (2012) reported that tactic based on analysis of error remained the main aspect in teaching and learning differential equations, where student incorporate concepts in a more successful way and performed better. However, students who were habituated with passive memorization found difficulties during detection of error.

The employment of meta-cognitive tools such as vee diagrams and concept maps were also introduced as innovative ways though which students' mathematics learning can be improved beyond their technical proficiency in applying known procedures and algorithms(Afamasaga-Fuata'i, 2002, 2003a, 2003b). These studies (vee diagrams and concept maps) were also extended to secondary level students in mathematical problem solving(Afamasaga-Fuata'i, 1998; Afamasaga-Fuata'i, 1999)and for undergraduate students (Afamasaga-Fuata'i, 1999, 2001, 2002). Afamasaga-Fuata'i (2004) evidently proved that student understanding of differential equation became more structured, well organized by using concept map and also it greatly assisted in the construction of vee diagram.

More recently, concept of reduced abstraction frame work is also introduced. Using this concept, Raychaudhuri (2013b) classified three learning levels of individual learner, in the context of DEs, on the basis of their effort maintain and reconstruct their cognitive structure. These are learners, potential learners and pseudo-learners. Authors elaborated that learner identifies and resolves the conflicts without discarding the existing connections between the concepts (Raychaudhuri, 2013b).Same group has given concept of extended reduced abstraction framework that explains student construction of mathematical knowledge of differential equation. In this case, the main focus was the solution to system of differential equation, including first order liner, non-linear, and second order as well as the corresponding existence and uniqueness theorem relevant to them (Raychaudhuri, 2013a).

6. CONCLUSIONS:

The differential equations are used to inter-relate and solve a variety of daily life. Several approaches have been developed and more are being developed or modified, to make DEs course more effective and useful. In this study, articles from 1996 to 2015interrelated to teaching and learning of DEs have been critically reviewed and analyzed. Five main categories including, algebraic, graphical, numerical, technological and inquiry oriented based approaches have been identified and compared.

Algebraic based approaches are traditional and are based on the several steps or procedures to solve the DEs analytically, while graphical methods are the qualitative in nature, and used to analyze either a graph or direction fields to solve a DE problem. In algebraic approach, focus is given to problem solving steps while numerical and graphical approaches are more emphasized to facilitate the conceptual learning of differential equations.

The reform moment has focused the on the incorporation and utilization of technological tools to integrate the traditional methods. These kinds of methods are attractive at higher levels of studies. However, at pre-university level, these are quite challenging due to the unavailability and interaction of these tools. An another era of reform moment is Inquiry oriented based approaches, which have further added the positive effect of environment, epistemological and motivational beliefs on the DEs learning.

Role of epistemological beliefs, motivational beliefs and self-regulating learning strategies were found effective, individually. Combination of these belief systems can be used to facilitate conceptual understanding and constructivist learning of differential equation including novel pedagogies (e.g. collaborative learning, inquiry/discovery), particularly at the college and pre university levels.

REFERENCES:

1. Abdulwahed, M., Jaworski, B., & Crawford, A. (2012). Innovative approaches to teaching mathematics in higher education: a review and critique. *Nordic Studies in Mathematics Education (NOMAD), 17*(2).

2. Afamasaga-Fuata'i, K. (1998). *Learning to Solve Mathematics Problems Through Concept Mapping & Vee Mapping: A Study of Form Five Students at Samoa College in Samoa*. Samoa: National University of Samoa.
3. Afamasaga-Fuata'i, K. (1999). Teaching mathematics and science using the strategies of concept mapping and Vee mapping. *Problems, Research, and Issues in Science, Mathematics, Computing and Statistics, 2*(1), 1-53.
4. Afamasaga-Fuata'i, K. (2001). *Enhancing students' understanding of mathematics using concept maps & Vee diagrams.* Paper presented at the International Conference on Mathematics Education (ICME), Northeast Normal University of China, Changchun, China.
5. Afamasaga-Fuata'i, K. (2002). *Vee diagrams & concept maps in mathematics problem solving.* Paper presented at the Pacific Education Conference (PEC), American Samoa Department of Education, American Samoa, Pago Pago.
6. Afamasaga-Fuata'i, K. (2003a). *Mathematics Education: Is it heading forward or backward.* Paper presented at the Measina Conference, Institute of Samoan Studies, National University of Samoa.
7. Afamasaga-Fuata'i, K. (2003b). *Numeracy in Samoa: From Trends & Concerns to Strategies.* Paper presented at the A Paper Presented at the Samoa's Principal Conference, Department of Education, Samoa, EFKS Hall.
8. Afamasaga-Fuata'i, K. (2004). *An undergraduate student's understanding of differential equations through concept maps and vee diagrams.* Paper presented at the Proceedings of the First International Conference on Concept Mapping, Pamplona, Spain.
9. Alpaslan, M. M., Yalvac, B., Loving, C. C., & Willson, V. (2016). Exploring the Relationship Between High School Students' Physics-Related Personal Epistemologies and Self-regulated Learning in Turkey. *International Journal of Science and Mathematics Education, 14*(2), 297-317. doi: 10.1007/s10763-015-9685-7
10. Aravind, P., Valluvan, M., & Ranganathan, S. (2013). Modelling and Simulation of Non Linear Tank. *International Journal of Advanced Research in Electrical, Electronics and Instrumentation Engineering, 2*(2), 842-849.
11. Arslan, S. (2010a). Do students really understand what an ordinary differential equation is? *International Journal of Mathematical Education in Science and Technology, 41*(7), 873-888. doi: 10.1080/0020739X.2010.486448
12. Arslan, S. (2010b). Traditional instruction of differential equations and conceptual learning. *Teaching Mathematics and its Applications, 29*(2), 94-107.
13. Artigue, M. (1989). A research of didactic engineering on the teaching of differential equations in Undergraduate Teaching in the Team Mathematics. *Research in Mathematics Education, 9*(3), 281-308.
14. Azman, A., & Ismail, Z. (2013). *Learning differential equations: A meta synthesis of qualitative research*. Paper presented at the International Seminar on Quality and Affordable Education, Johor, Malaysia
15. Blanchard, P. (1994). Teaching differential equations with a dynamical systems viewpoint. *The college mathematics journal, 25*(5), 372-384.
16. Blanchard, P., Devaney, R., & Hall, G. (2002). *Differential equations* (2nd ed.). MARK McCARTNEY and SHARON GIBSON School of Computing and Mathematics, University of Ulster, Shore Road, Newtownabbey, Northern Ireland.
17. Borrelli, R., & Coleman, C. (1999). *Modeling and visualization in the introductory ODE course.* Paper presented at the Mathematical Association of America, Washington,. D. C.
18. Borrelli, R. L., & Coleman, C. S. (1996). *Student Solutions Manual to Accompany Differential Equations: A Modeling Perspective* (2nd ed.). Corporate Headquarters, 111 River Street: Wiley, USA.
19. Boyce, W. E. (1994). New directions in elementary differential equations. *College Mathematics Journal, 25*(5), 364-371.
20. Boyce, W. E. (1999). *Differential equations in the information age*. Paper presented at the Mathematical Association of America, Washington, D. C.
21. Bråten, I., & Strømsø, H. I. (2005). The relationship between epistemological beliefs, implicit theories of intelligence, and self- regulated learning among Norwegian postsecondary students. *British Journal of Educational Psychology, 75*(4), 539-565.
22. Bybee, R. (2001). Constructivism and the Five E's. Miami Museum of Science. http://www.miamisci.org/ph/lpintro5e.html
23. Camacho-Machín, M., & Guerrero-Ortiz, C. (2015). Identifying and exploring relationships between contextual situations and ordinary differential equations. *International Journal of Mathematical Education in Science and Technology, 46*(8), 1077-1095.
24. Camacho-Machín, M., Perdomo-Díaz, J., & Santos-Trigo, M. (2008). Revisiting university students'knowledge that involves basic differential equation questions. *International Group for the Psychology of Mathematics Education, 3*(3), 123-133.
25. Camacho-Machín, M., Perdomo-Díaz, J., & Santos-Trigo, M. (2012a). An exploration of students' conceptual knowledge built in a first ordinary differential equations course (Part I). *The Teaching of Mathematics, XV, 1*, 1-20.

26. Camacho-Machín, M., Perdomo-Díaz, J., & Santos-Trigo, M. (2012b). An exploration of students' conceptual knowledge built in a first ordinary differential equations course (Part II). *Teaching of Mathematics, 15*(2), 63-84.
27. Cobb, P. (1985). Two children's anticipations, beliefs, and motivations. *Educational Studies in Mathematics, 16*(2), 111-126.
28. Cobb, P. (1995). Cultural tools and mathematical learning: A case study. *Journal for research in mathematics education, 26*(4), 362-385.
29. Coombes, K. R., Stuck, G. J., Lipsman, R. L., Osborn, J. E., & Hunt, B. R. (1995). *Differential equations with Maple* (Vol. 2nd): John Wiley & Sons, Inc. New York, NY, USA.
30. Czocher, J. A., Tague, J., & Baker, G. (2013). Where does the calculus go? An investigation of how calculus ideas are used in later coursework. *International Journal of Mathematical Education in Science and Technology, 44*(5), 673-684.
31. Dana-Picard, T., & Kidron, I. (2008). Exploring the phase space of a system of differential equations: different mathematical registers. *International Journal of Science and Mathematics Education, 6*(4), 695-717. doi: 10.1007/s10763-007-9099-2
32. Dawkins, P. C., & Epperson, J. A. M. (2014). The development and nature of problem-solving among first-semester calculus students. *International Journal of Mathematical Education in Science and Technology, 45*(6), 839-862.
33. Devaney, R. L., West, B., Strogatz, S., McDill, J. M., & Cantwell, J. (1998). Interactive Differential Equations. *American Mathematical Monthly, 105*(7), 687-689.
34. Fadlelmula, F. K., Cakiroglu, E., & Sungur, S. (2015). Developing a Structural Model on the Relationship Among Motivational Beliefs, Self-Regulated Learning Strategies, and Achievement in Mathematics. *International Journal of Science and Mathematics Education, 13*(6), 1355-1375. doi: 10.1007/s10763-013-9499-4
35. Gado, I. (2005). Determinants of K-2 School Teachers' Orientation Towards Inquiry-Based Science Activities: A Mixed Method Study. *International Journal of Science and Mathematics Education, 3*(4), 511-539. doi: 10.1007/s10763-005-0689-6
36. Gollwitzer, H. (1991). *Visualization in differential equations.* Paper presented at the Visualization in teaching and learning mathematics, Washington, DC, USA.
37. Habre, S. (2000). Exploring students' strategies to solve ordinary differential equations in a reformed setting. *The Journal of Mathematical Behavior, 18*(4), 455-472.
38. Habre, S. (2002). *Writing in a Reformed Differential Equations Class.* Paper presented at the International Conference on the Teaching of Mathematics, Greece.
39. Habre, S. (2003). Investigating students' approval of a geometrical approach to differential equations and their solutions. *International Journal of Mathematical Education in Science and Technology, 34*(5), 651-662.
40. Hofer, B. K. (1999). Instructional context in the college mathematics classroom: Epistemological beliefs and student motivation. *Journal of Staff, Program & Organization Development, 16*(2), 73-82.
41. Holton, D. (2001). *The teaching and learning of mathematics at university level: An ICMI study* (Vol. 7). Dordrecht, The Netherlands: Kluwer.
42. Hu, D., & Rebello, N. S. (2013). Understanding student use of differentials in physics integration problems. *Physical Review Special Topics-Physics Education Research, 9*(2), 1-14.
43. Hubbard, J. H., & West, B. H. (2012). *Differential equations: a dynamical systems approach: higher-dimensional systems* (Vol. 18). New York, NY, USA: Springer-Verlag Inc.
44. Ju, M.-K., & Kwon, O. N. (2004). Analysis of students' use of metaphor: The case of a RME-based differential equations course. *Journal of the Korea Society of Mathematical Education, 8*(1), 19-30.
45. Ju, M.-K., & Kwon, O. N. (2007). Ways of talking and ways of positioning: Students' beliefs in an inquiry-oriented differential equations class. *The Journal of Mathematical Behavior, 26*(3), 267-280. doi: http://dx.doi.org/10.1016/j.jmathb.2007.10.002
46. Keene, K. A., Glass, M., & Kim, J. H. (2011). *Identifying and assessing relational understanding in ordinary differential equations.* Paper presented at the Frontiers in Education Conference (FIE), 2011, Rapid City, SD, USA
47. Kim, M., & Kwon, O. (2006). Students' argumentation and mathematical justification in inquiry-oriented learning. *Unpublished master thesis, Seoul National University, Seoul, Korea*.
48. Kingir, S., Tas, Y., Gok, G., & Vural, S. S. (2013). Relationships among constructivist learning environment perceptions, motivational beliefs, self-regulation and science achievement. *Research in Science & Technological Education, 31*(3), 205-226.
49. Kostelich, E. J., & Armbruster, D. (1996). *Introductory differential equations: From linearity to chaos* (1st ed.). Boston MA 02116 USA: Addison Wesley.
50. Kwon, O. N. (2002). *Conceptualizing the Realistic Mathematics Education Approach in the Teaching and Learning of Ordinary Differential Equations.* Paper presented at the International Conference on the Teaching of Mathematics (at the Undergraduate Level) 2nd, Hersonissos, Crete, Greece.
51. Kwon, O. N., Rasmussen, C., & Allen, K. (2005). Students' Retention of Mathematical Knowledge and Skills in Differential Equations. *School science and mathematics, 105*(5), 227-239.
52. Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. New York, NY, USA: Cambridge University Press.
53. Lomen, D., & Lovelock, D. (1996). *Exploring differential equations via graphics and data*: John Wiley & Sons Inc.

54. Machín, M. C., Díaz, J. P., & Trigo, L. M. S. (2009). Revisiting university students' knowledge that involves basic differential equation questions. *PNA, 3*(3), 123-133.
55. Mallet, D. G., & McCue, S. W. (2009). Constructive development of the solutions of linear equations in introductory ordinary differential equations. *International Journal of Mathematical Education in Science and Technology, 40*(5), 587-595.
56. Meier, J., & Rishel, T. (1998). *Writing in the Teaching and Learning of Mathematics*. New York, NY, USA: Cambridge University Press.
57. Montalvo, F. T., & Torres, M. C. G. (2004). Self-regulated learning: Current and future directions. *Electronic journal of research in educational psychology, 2*(1), 1-34.
58. Moore, T. J., Miller, R. L., Lesh, R. A., Stohlmann, M. S., & Kim, Y. R. (2013). Modeling in engineering: The role of representational fluency in students' conceptual understanding. *Journal of Engineering Education, 102*(1), 141-178.
59. Muis, K. R. (2007). The role of epistemic beliefs in self-regulated learning. *Educational Psychologist, 42*(3), 173-190.
60. Muis, K. R. (2008). Epistemic profiles and self-regulated learning: Examining relations in the context of mathematics problem solving. *Contemporary Educational Psychology, 33*(2), 177-208.
61. Muis, K. R., & Franco, G. M. (2009). Epistemic beliefs: Setting the standards for self-regulated learning. *Contemporary Educational Psychology, 34*(4), 306-318.
62. Perels, F., Gürtler, T., & Schmitz, B. (2005). Training of self-regulatory and problem-solving competence. *Learning and Instruction, 15*(2), 123-139.
63. Pollak, H. O. (2015). The Place of Mathematical Modelling in the System of Mathematics Education: Perspective and Prospect *Mathematical Modelling in Education Research and Practice* (pp. 265-276). Gewerbestrasse 11, CH-6330 Cham (ZG), Switzerland: Springer International Publishing.
64. Polya, G. (1981). Mathematical discovery: On understanding, learning, and teaching problem solving,(Combined Edition). *New York, John Wiley & Sons. Baillie R., Borwein D., and Borwein J.(2008),"Some sinc sums and integrals," American Math. Monthly, 115*(10), 888-901.
65. Porter, M. K., & Masingila, J. O. (1995). *The effects of writing to learn mathematics on the types of errors students make in a college calculus class.* Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, 17th, Columbus, OH.
66. Rasmussen, C., & Blumenfeld, H. (2007a). Reinventing solutions to systems of linear differential equations: A case of emergent models involving analytic expressions. *The Journal of Mathematical Behavior, 26*(3), 195-210.
67. Rasmussen, C., & Kwon, O. N. (2007b). An inquiry-oriented approach to undergraduate mathematics. *The Journal of Mathematical Behavior, 26*(3), 189-194.
68. Rasmussen, C., Kwon, O. N., Allen, K., Marrongelle, K., & Burtch, M. (2006a). Capitalizing on advances in mathematics and K-12 mathematics education in undergraduate mathematics: An inquiry-oriented approach to differential equations. *Asia Pacific Education Review, 7*(1), 85-93.
69. Rasmussen, C., & Marrongelle, K. (2006b). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education, 37*(5), 388-420.
70. Rasmussen, C. L. (1996). *Qualitative problem solving strategies of first order differential equations: the case of Amy.* Paper presented at the Electronic Proceedings of the Fifth Conference on the Teaching of Mathematics, Baltimore, Maryland.
71. Rasmussen, C. L. (1998). *Reform in Differential Equations: A Case Study of Students' Understandings and Difficulties.* Paper presented at the Annual Meeting of the American Educational Research Association, San Diego, CA.
72. Rasmussen, C. L. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *The Journal of Mathematical Behavior, 20*(1), 55-87.
73. Rasmussen, C. L., & King, K. D. (2000). Locating starting points in differential equations: A realistic mathematics education approach. *International Journal of Mathematical Education in Science and Technology, 31*(2), 161-172.
74. Raychaudhuri, D. (2008). Dynamics of a definition: a framework to analyse student construction of the concept of solution to a differential equation. *International Journal of Mathematical Education in Science and Technology, 39*(2), 161-177. doi: 10.1080/00207390701576874
75. Raychaudhuri, D. (2013a). Adaptation and extension of the framework of reducing abstraction in the case of differential equations. *International Journal of Mathematical Education in Science and Technology, 45*(1), 35-57. doi: 10.1080/0020739X.2013.790503
76. Raychaudhuri, D. (2013b). A framework to categorize students as learners based on their cognitive practices while learning differential equations and related concepts. *International Journal of Mathematical Education in Science and Technology, 44*(8), 1239-1256.
77. Rehman, A.-u., & Masud, T. (2012). *Calculus and Analytic Geometry, Mathematics 12*. khyber pakhtunkhwa,Pakistan: Abbottabad Text Book Board
78. Rogoff, B. (1990). *Apprenticeship in thinking: Cognitive development in social context*. New York, NY, USA: Oxford University Press.
79. Rowland, D. R. (2006). Student difficulties with units in differential equations in modelling contexts. *International Journal of Mathematical Education in Science and Technology, 37*(5), 553-558.

80. Rowland, D. R., & Jovanoski, Z. (2004). Student interpretations of the terms in first-order ordinary differential equations in modelling contexts. *International Journal of Mathematical Education in Science and Technology, 35*(4), 503-516.
81. Sánchez, J. J. B. (2012). *The analysis of errors in the solution of ordinary differential equations* Paper presented at the Proceedings in Advanced Research in Scientific Areas (ARSA), Bratislava, Slovakia.
82. Saxe, L. (1991). Lying: Thoughts of an applied social psychologist. *American Psychologist, 46*(4), 409-415.
83. Schoenfeld, A. H. (1983). Beyond the purely cognitive: Belief systems, social cognitions, and metacognitions as driving forces in intellectual performance. *Cognitive science, 7*(4), 329-363.
84. Schoenfeld, A. H. (1985). Metacognitive and epistemological issues in mathematical understanding. *Teaching and learning mathematical problem solving: Multiple research perspectives, 89*(4), 361-380.
85. Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for research in mathematics education, 20*(4), 338-355.
86. Schommer-Aikins, M. (2004). Explaining the epistemological belief system: Introducing the embedded systemic model and coordinated research approach. *Educational psychologist, 39*(1), 19-29.
87. Stockton, J. C. (2010). *A study of the relationships between epistemological beliefs and self-regulated learning among advanced placement calculus students in the context of mathematical problem solving.* (Dissertation), Kennesaw State University, Kennesaw, Georgia.
88. Velayutham, S., Aldridge, J. M., & Fraser, B. (2012). Gender differences in student motivation and self-regulation in science learning: A multi-group structural equation modeling analysis. *International Journal of Science and Mathematics Education, 10*(6), 1347-1368. doi: 10.1007/s10763-012-9339-y
89. Wagner, J. F., Speer, N. M., & Rossa, B. (2007). Beyond mathematical content knowledge: A mathematician's knowledge needed for teaching an inquiry-oriented differential equations course. *The Journal of Mathematical Behavior, 26*(3), 247-266. doi: http://dx.doi.org/10.1016/j.jmathb.2007.09.002
90. West, B. H. (1996). *Interactive differential equations workbook*: Addison Wesley.
91. Wolters, C. A., Shirley, L. Y., & Pintrich, P. R. (1996). The relation between goal orientation and students' motivational beliefs and self-regulated learning. *Learning and individual differences, 8*(3), 211-238.
92. Yackel, E., Rasmussen, C., & King, K. (2000). Social and sociomathematical norms in an advanced undergraduate mathematics course. *The Journal of Mathematical Behavior, 19*(3), 275-287.
93. Yurtseven Avci, Z., Vasu, E. S., Oliver, K., Keene, K. A., & Fusarelli, B. (2014). Utilization of online technologies in mathematical problem solving at high school level: Student and teacher perceptions. *World Journal on Educational Technology, 6*(2), 1-23.
94. Zandieh, M., & McDonald, M. (1999). *Student understanding of equilibrium solution in differential equations.* Paper presented at the Proceedings of the 21st annual meeting of the North American chapter of the international group for the psychology of mathematics education (pp. 253-258), Columbus, ERIC.