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A Mid-Width Method for Time Minimizing Transshipment Problem with Interval Constraints

Himanshu Agrawal [*1], Shambhu Sharma [2]
Department of Mathematics, Dayalbagh Educational Institute, Dayalbagh, Agra-282005, India
Email - [1]himanshuagrawal74143@gmail.com [2]shambhusharma@dei.ac.in

***Abstract**: This paper presents a method for solving a variant of time minimizing transshipment problem containing intervals for time and constraints. The goal is to find a least transportation time when moving a single type of product from suppliers to customers. In this article, first, the time minimizing transshipment problem of interval constraints is converted into a time-minimizing transportation problem of interval constraints by adding buffer stock at each transshipment point. Then, the mid-width method is applied to obtain the feasible solution. Using mid-width method, the transportation problem decomposed into two different crisp transportation problems, a mid-value transportation problem and a half-width transportation problem. The mid-value transportation problem is solved using shootout method to find the optimal solution. The half-width transportation problem is solved starting with the basic cells to be same with the mid-value transportation problem. The solutions to these decomposed problems are then integrated to produce the solution for the transformed transportation problem, ultimately leading to the solution of the original transshipment problem. The proposed algorithm demonstrates the solution procedure and is justified with the help of a numerical illustration.*

***Keywords**: Time minimizing Transportation Problem; Transshipment Problem; Time Minimizing Transshipment Problem; buffer stock; Interval transportation problem; Interval transshipment problem.*

1. INTRODUCTION :

The Transportation Problem (TP) is a type of linear programming problem that involves finding the most cost-effective shipping routes to transport a single type of product from multiple sources to multiple destinations while meeting their supply and demand requirements. This problem was first introduced by Hitchcock [2]. The problem of minimizing the duration of transportation was also studied by many researchers. In a time minimizing transportation problem (TMTP), the goal is to transport a homogeneous product from sources to destinations in least possible time while satisfying the supply and demand constraints. TMTP has been extensively studied by many researchers Hammer [4], Szwarc [5] Khurana and Arora [6], Bhatia et al. [7], Prakash [9]. Hammer [4] and Szwarc [5] used leveling techniques to solve this problem. Agrawal and Sharma [10-13] introduced time minimizing transportation problem with mixed constraints (TMTP-MC). They have developed various methods to solve it, such as the open loop method [10], minimax method [11], shootout method [12] and also examined the MFL paradoxical situation [13]. Their methods can also be applied to the TMTP.

Orden [3] extended the transportation problem by allowing for transshipment, where any shipping or receiving point can also act as an intermediate point. This helps to lower transportation costs and save time compared to not using transshipment. The problem of minimizing the time in transshipment problem was first introduced by Garg and Prakash [9], known as the time minimizing transshipment problem (TMTsP). They developed a two-phase computational method to obtain the solution of time minimizing transshipment problem. Further, Khurana and Verma [8] established a method to solve TMTsP keeping in mind the objective to minimize the maximum duration of time in transportation. In this method they solved the transshipment problem by transforming it into an equivalent transportation problem. Optimal solution of the transshipment problem is obtained from the optimal solution of the transformed transportation problem. In real-life situations, supply, demand and transportation cost can vary within a specific range rather than being fixed. Dealing with this variability, known as the Transportation Problem with Interval Constraints, is essential for practical applications. Interval constraints allow for a more realistic representation of supply and demand fluctuations, leading to more robust solutions that can be applied to everyday scenarios. Several well-organized techniques for solving transportation problems with interval source, destination parameter, and cost were established [14-19]. Sengupta and Pal [18] introduced a new fuzzy orientation approach to solve interval transportation problems by considering the midpoint and width of the interval in the objective function. Pandian and Natarajan [14] developed the separation method to solve integer interval transportation problems and minimize costs using the zero-point method. Subhakanta, Dash, and Mohanty [21] discussed a method for solving transportation problems that considers the unit cost of transportation from a source to a destination as a rough integer interval. Akilbasha et al. [14] proposed the split and separation method to find optimal solutions for integer transportation problems in a rough environment. Pandian et al. [15] developed the slice-sum method to determine optimal solutions for fully rough interval integer transportation problems. Akilbasha et al. [22] introduced an innovative exact method for solving fully interval integer transportation problems. The mid-width method is used to find optimal solutions for interval transportation problems where shipping costs, supply, and demand parameters are real intervals. This method is exact and is based on two independent transportation problems derived from a fully integer transportation problem. This concept inspired the development of the proposed methodology.

P. Rajendran and P. Pandian [24] proposed a method namely, splitting method for finding an optimal solution to a transshipment problem in which all parameters are real intervals. Porchelvi, S. et al. [23] gave a comparative study between optimum solution of transportation and transshipment problem for the cost minimizing objective function where the source, destination parameters and cost is expressed as interval values. However, the Time Minimizing Transshipment Problem with interval Constraints (TMTsP-IC) has not been studied yet.

This paper presents the mid-width method to find a basic feasible solution for the TMTsP-IC. Firstly, the problem is transformed into a time-minimizing transportation problem by adding buffer stock at each transshipment point. The mid-width method then decomposes the problem into two distinct transportation problems: a mid-value transportation problem, and a half-width transportation problem. The mid-value transportation problem is solved using the shootout method [12] and find an optimal solution. The half-width transportation problem is solved with initial basic cells identical to the mid-value problem. The solutions to these decomposed problems are combined to resolve the original transshipment problem. The organization of the paper is as follows. In section 2 preliminary definition are given. In Section 3, the mathematical formulation of TMTsP-IC is given. In section 4, the theoretical development of the proposed algorithm for TMTsP and its stepwise procedure are described in section 5. Section 6 consists of a numerical illustration that helps to verify the working of the algorithm. Section 7 is dedicated to conclusion.

2. Preliminaries :

Some fundamental definitions and results related to real intervals are as follows

Let $A = [a_L, a_R] = \{a: a_L \le a \le a_R, a \in \mathbb{R}\}$, where a_L and a_R are the left and right limits of A, respectively. The interval is also denoted by its mid-value (center) and half-width as A= $\langle a_m, a_w \rangle = \{a: a_m - a_w \le a \le a_m + a_w, a \in \mathbb{R}\}$, where $a_m = (a_R + a_L)/2$ is the center or mid-value of A and $a_w = (a_R - a_L)/2$ is the half-width of A.

Algebra of intervals: Let $A = [a_L, a_R] = \{a: a_L \le a \le a_R, a \in \mathbb{R}\}$ and $B = [b_L, b_R] = \{x: b_L \le x \le b_R, a \in \mathbb{R}\}$ be two intervals then

i. $A + B = [a_L + b_L, a_R + b_R]$ and
ii. $kA = [ka_L, ka_R]$ *where k is positive number*
iii. $kA = [ka_R, ka_L]$ *where k is negative number*
iv. $A \times B = [p, q]$,wher p=minimum $\{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}$ and q= maximum $\{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}$

Definition 1. Let $A = [a_L, a_R]$ and $B = [b_L, b_R]$ be two intervals. Then the order relation between A and B is defined as, $A \le B$ if $a_L \le b_L$ and $a_R \le b_R$, $A = B$ if and only if $a_L = b_L$ and $a_R = b_R$

Definition 2. Let $A = [a_L, a_R]$ be real interval. Then, A is said to be positive denoted by $A \ge 0$ if $a_L \ge 0$

Definition 3. Let $A = [a_L, a_R]$ be interval. Then, A is said to be integer interval if a_L and a_R are the integers.

Definition 4. Let $A = [a_L, a_R]$ and $B = [b_L, b_R]$ be two intervals. The Ordering of two intervals based on mid-value is as follow,

(i) $A = B$ if $m(A) = m(\text{B})$
(ii) $A > B$ if $m(A) > m(\text{B})$
(iii) $A < B$ if $m(A) < m(\text{B})$

Definition 5. Let $A = [a_L, a_R]$ and $B = [b_L, b_R]$ be two intervals. The ordering of two intervals based on half width is as follow,

(i) $A = B$ if $w(A) = w(B)$
(ii) $A \ge B$ if $w(A) \ge w(B)$
(iii) $A \le B$ if $w(A) \le w(B)$

Result: if $m(A) = m_1$ and $w(A) = w_1$ $then$ $A = [m_1 - w_1, m_1 + w_1]$

Interval Ordering: Ordering real intervals in mathematics can be approached in different ways depending on the context and the criteria for ordering. Here are a few common methods:

Lexicographic Order: This method involves ordering intervals based on their endpoints. First, compare the left endpoints of the intervals. If the left endpoints are equal, then compare the right endpoints. Example: Consider the set of intervals, S= {[1, 3], [2, 4], [1, 2]}. Lexicographically, they would be ordered as: [1, 2], [1, 3], [2, 4]. Min S= [1,2], Max S= [2,4],

Order by Length: Intervals can be ordered by their length (i.e., the difference between the right and left endpoints). This method can order intervals from the shortest to the longest or vice versa. Example: Consider the set S= {[1, 3], [2, 4], [1, 2]}. The lengths of intervals of S are 2, 2, and 1, respectively. Ordered by length, they would be: [1, 2], [1, 3], [2, 4]. Min S= [1,2], Max S= [2,4],

Order by Left Endpoint: Intervals can be ordered by their left endpoint only. If two intervals have the same left endpoint, they can be further ordered by the right endpoint as a secondary criterion. Example: Consider the set of intervals S. Ordered by the left endpoint, they would be: [1, 2], [1, 3], [2, 4]. Min S= [1,2], Max S= [2,4],

Containment Order: This method orders intervals based on whether one interval is contained within another. An interval [a, b] is less than or equal to an interval [c, d] if [a, b] sub set [c, d]. Example: Consider the set of intervals S. Ordered by containment, [1, 2] sub set [1, 3], so the order could be [1, 2], [1, 3], and [2, 4] (since there is no containment relationship between [1, 3] and [2, 4].

Custom Order: Sometimes intervals are ordered based on specific custom criteria relevant to a particular problem or application. For example, intervals might be ordered based on their midpoints, or based on a function applied to their endpoints. Example: Suppose we order intervals by the midpoint of the interval. Consider the set of intervals S. The midpoints are 2, 3, and 1.5, respectively. Ordered by midpoint, they would be: [1, 2], [1, 3], [2, 4]. Min S= [1,2], Max S= [2,4].

Remark: The choice of ordering method depends on the specific requirements and context of the problem at hand. Each method provides a different perspective on how to compare and arrange intervals.

For more information related to intervals ordering see [14-20].

3. Mathematical Model of The Transshipment Problem with Interval Constraints (TMTsP-IC) :

Suppose that, we have 'm' sources and 'n' destinations. Since, in a transshipment problem, any source or destination can ship to any other source or destination so it would be convenient to number them successively. The sources are numbered from 1 to m and the destinations are numbered from m to $m + n$. Let $\left[a_{L_i}, a_{R_i}\right]$ be the quantities range available at the origins O_i and $\left[b_{L_j}, b_{R_j}\right]$ be the range of demand at the destinations D_j. In the transshipment problem $O_1, O_2, \dots O_i, \dots O_m$ are sources from where a homogeneous commodity is transported to destinations $D_{m+1}, D_{m+2}, \dots D_{m+j}, \dots D_{m+n}$. Now each point $O_1, O_2, \dots O_i, \dots O_m D_{m+1}, D_{m+2}, \dots D_{m+j}, \dots\ D_{m+n}$ behaves like supply as well as demand point so called as transshipment point.

Let $[x_{ij}, y_{ij}]$ be the quantity range shipped from transshipment point $O_i; i = 1,2 \dots m, D_j; j = m+1, m+2, \dots m+n$ to the transshipment point $O_i; i = 1,2 \dots m, D_j; j = m+1, m+2, \dots m+n$ and $[t_{L_{ij}}, t_{R_{ij}}]$ be the corresponding shipping time range. The table-1 is demonstrated for interpreting the given information which allows a compact representation of the problem.

In time minimizing transshipment problem with interval constraints we have to find $[t_{L_{ij}}, t_{R_{ij}}]$ and $[x_{ij}, y_{ij}]$ which

(P) $$Minimize\ Z = \max_{i,j} \left\{[t_{L_{ij}}, t_{R_{ij}}] : [x_{ij}, y_{ij}] > 0\right\}$$

Subject to,

$$\sum_{j=1}^{m+n} [x_{ij}, y_{ij}] - \sum_{j=1}^{m+n} [x_{ji}, y_{ji}] = [a_{L_i}, a_{R_i}], i = 1,2,3, \dots, m \qquad (1)$$

$$\sum_{i=1}^{m+n} [x_{ij}, y_{ij}] - \sum_{i=1}^{m+n} [x_{ji}, y_{ji}] = [b_{L_j}, b_{R_j}], j = 1,2,3, \dots, n \qquad (2)$$

$$[x_{ij}, y_{ij}] \geq 0\ ; i, j = 1,2, \dots m+n, i \neq j$$

$$[t_{L_{ii}}, t_{R_{ii}}] = 0, i = 1,2, \dots m+n$$

$$[a_{L_i}, a_{R_i}] \geq 0\ ; i = 1,2 \dots m\ and\ [b_{L_j}, b_{R_j}] \geq 0\ ; j = m+1, m+2 \dots m+n.$$

Unbalanced transshipment problem arises when

$$\sum_{i=1}^{m}[a_{L_i}, a_{R_i}] \neq \sum_{j=m+1}^{m+n}[b_{L_j}, b_{R_j}]$$

Note that in transshipment problem $[x_{ii}, y_{ii}] = \alpha$, does not make any prolific sense as it represents the quantity shipped from the i^{th}-transshipment point to the same transshipment point so in the final feasible solution of the transshipment problem this can be avoided.

It is assumed in the proposed problem that

i. The carriers have sufficient capacity to carry goods from source to a destination in a single trip
ii. They start simultaneously from their respective sources

Let us make note of the following definitions which will be useful in the paper.

1. ***Feasible solution:*** The set $\boldsymbol{S} = \{[\boldsymbol{x_{ij}}, \boldsymbol{y_{ij}}]: [\boldsymbol{x_{ij}}, \boldsymbol{y_{ij}}] \geq \boldsymbol{0}\}$ is said to be a feasible solution of TMTsP-IC (P) if it satisfies the equations (1) and (2). $\boldsymbol{T} = [\boldsymbol{t_{L_{\alpha\beta}}}, \boldsymbol{t_{R_{\alpha\beta}}}] = \max_{i,j}\{[\boldsymbol{t_{L_{ij}}}, \boldsymbol{t_{R_{ij}}}] : [\boldsymbol{x_{ij}}, \boldsymbol{y_{ij}}] > \boldsymbol{0}\}$ is the time span corresponding to the feasible solution S.

2. ***Better Feasible solution***: Let $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ be two feasible solution of (P) and $\boldsymbol{T_1}$ ***and*** $\boldsymbol{T_2}$ be the corresponding time span then $\boldsymbol{S_2}$ is said to be a better feasible solution than $\boldsymbol{S_1}$, if $\boldsymbol{T_2} < \boldsymbol{T_1}$.

3. ***Optimal solution:*** A feasible solution is said to be an optimal solution if there does not exist any better feasible solution, i.e. for which the time of transportation is the least.

Table-1: *Table of TMTsP-IC*

	O_1	…	O_i	…	O_m	D_{m+1}	…	D_{m+j}	…	D_{m+n}	*supply*
O_1	$[t_{L_{11}}, t_{R_{11}}]$	…	$[t_{L_{1i}}, t_{R_{1i}}]$	…	$[t_{L_{1,m}}, t_{R_{1,m}}]$	$[t_{L_{1,m+1}}, t_{R_{1,m+1}}]$	…	$[t_{L_{1,m+j}}, t_{R_{1,m+j}}]$	…	$[t_{L_{1,m+n}}, t_{R_{1,m+n}}]$	$[a_{L_1}, a_{R_1}]$
O_2	$[t_{L_{21}}, t_{R_{21}}]$	…	$[t_{L_{2i}}, t_{R_{2i}}]$	…	$[t_{L_{2,m}}, t_{R_{2,m}}]$	$[t_{L_{2,m+1}}, t_{R_{2,m+1}}]$	…	$[t_{L_{2,m+j}}, t_{R_{2,m+j}}]$	…	$[t_{L_{2,m+n}}, t_{R_{2,m+n}}]$	$[a_{L_2}, a_{R_2}]$
⋮	⋮	⋱	⋮	⋱	⋮	⋮	⋱	⋮	⋱	⋮	⋮
O_i	$[t_{L_{i1}}, t_{R_{i1}}]$	…	$[t_{L_{ii}}, t_{R_{ii}}]$	…	$[t_{L_{i,m}}, t_{R_{i,m}}]$	$[t_{L_{i,m+1}}, t_{R_{i,m+1}}]$	…	$[t_{L_{i,m+j}}, t_{R_{i,m+j}}]$	…	$[t_{L_{i,m+n}}, t_{R_{i,m+n}}]$	$[a_{L_i}, a_{R_i}]$
⋮	⋮	⋱	⋮	⋱	⋮	⋮	⋱	⋮	⋱	⋮	⋮
O_m	$[t_{L_{m,1}}, t_{R_{m,1}}]$	…	$[t_{L_{m,i}}, t_{R_{m,i}}]$	…	$[t_{L_{m,m}}, t_{R_{m,m}}]$	$[t_{L_{m,m+1}}, t_{R_{1,m+1}}]$	…	$[t_{L_{mm+j}}, t_{R_{mm+j}}]$	…	$[t_{L_{m,m+n}}, t_{R_{m,m+n}}]$	$[a_{L_m}, a_{R_m}$
D_{m+1}	$[t_{L_{m+1,1}}, t_{R_{m+1,1}}$	…	$[t_{L_{m+1,i}}, t_{R_{m+1,i}}$	…	$[t_{L_{m+1,m}}, t_{R_{m+1,m}}$	$[t_{L_{m+1,m+1}}, t_{R_{m+1,m+1}}$	…	$[t_{L_{m+1,m+j}}, t_{R_{m+1,m+j}}$	…	$[t_{L_{m+1,m+n}}, t_{R_{m+1,m+n}}$	–
D_{m+2}	$[t_{L_{m+2,1}}, t_{R_{m+2,1}}$	…	$[t_{L_{m+2,i}}, t_{R_{m+2,i}}$	…	$[t_{L_{m+2,m}}, t_{R_{m+2,m}}$	$[t_{L_{m+2,m+1}}, t_{R_{m+2,m+1}}$	…	$[t_{L_{m+2,m+j}}, t_{R_{m+2,m+j}}$	…	$[t_{L_{m+2,m+n}}, t_{R_{m+2,m+n}}$	–
⋮											⋮
D_{m+j}	$[t_{L_{m+j,1}}, t_{R_{m+j,1}}$	…	$[t_{L_{m+j,i}}, t_{R_{m+j,i}}$	…	$[t_{L_{m+j,m}}, t_{R_{m+j,m}}$	$[t_{L_{m+j,m+1}}, t_{R_{m+j,m+1}}$	…	$[t_{L_{m+j,m+j}}, t_{R_{m+j,m+j}}$	…	$[t_{L_{m+j,m+n}}, t_{R_{m+j,m+n}}$	–
⋮											⋮
D_{m+n}	$[t_{L_{m+n,1}}, t_{R_{m+n,1}}$	…	$[t_{L_{m+n,i}}, t_{R_{m+n,i}}$	…	$[t_{L_{m+n,m}}, t_{R_{m+n,m}}$	$[t_{L_{m+n,m+1}}, t_{R_{m+n,m+1}}$	…	$[t_{L_{m+n,m+j}}, t_{R_{m+n,m+j}}$	…	$[t_{L_{m+n,m+n}}, t_{R_{m+n,m+n}}$	–
demand	–	–	–	–	–	$[b_{L_{m+1}}, b_{R_{m+1}}]$	…	$[b_{L_{m+j}}, b_{R_{m+j}}]$	…	$[b_{L_{m+n}}, b_{R_{m+n}}]$	–

4. Methodology :

Transition from Time minimizing Transshipment problem with interval constraints to time minimizing transportation problem with interval constraints

A method for transforming the TMTsP into TMTP has been developed by Khurana and Arora [8]. Similarly, we transform the TMTsP-IC into the TMTP-IC. If we add buffer stock $[m_L, m_R] = \sum_{i=1}^{m}[a_{L_i}, a_{R_i}] = \sum_{j=m+1}^{m+n}[b_{L_j}, b_{R_j}]$ to each transshipment point, then the transshipment problem of $m \times n$ order can be converted to the transportation problem of $(m+n) \times (m+n)$ order, which is

(P₁) $Minimize\ Z = \max_{i,j}\{[\boldsymbol{t_{L_{ij}}}, \boldsymbol{t_{R_{ij}}}] : [x_{ij}, y_{ij}] > 0\}$

Subject to,

$$\sum_{j=1}^{m+n}[x_{ij},y_{ij}] = [a_{L_i}+m_L, a_{R_i}+m_R], i = 1,2,3,\dots,m \quad (3)$$

$$\sum_{j=1}^{m+n}[x_{ij},y_{ij}] = [m_L,m_R], i = m+1, m+2,\dots m+n \quad (4)$$

$$\sum_{i=1}^{m+n}[x_{ij},y_{ij}] = [m_L,m_R], j = 1,2,\dots m \quad (5)$$

$$\sum_{i=1}^{m+n}[x_{ij},x_{ij}] = \left[b_{L_j}+m_L, b_{R_j}+m_R\right], j = m+1, m+2,\dots m+n \quad (6)$$

$$[x_{ij},y_{ij}] \geq 0 ; i,j = 1,2,\dots m+n, i \neq j$$

$$[a_{L_j},a_{R_j}] \geq 0 ; i = 1,2\dots m \text{ and } [b_{L_j},b_{R_j}] \geq 0 ; j = m+1, m+2\dots m+n$$

Theorem: Feasible solution of TMTP-IC (P_1) is also feasible solution of TMTsP-IC (P).

Proof: let $S = \{[x_{ij},y_{ij}]: [x_{ij},y_{ij}] \geq 0\}$ be feasible solution of P_1, then it will hold (3), (4), (5) and (6) with non-negativity constraints.

Subtract each equation of (5) from corresponding equation of (3), we get

$$\sum_{j=1}^{m+n}[x_{ij},y_{ij}] - \sum_{j=1}^{m+n}[x_{ji},y_{ji}] = [a_{L_i},a_{R_i}], i = 1,2,3,\dots,m$$

Similarly, subtract each equation of (6) from corresponding equation of (4), we get

$$\sum_{i=1}^{m+n}[x_{ij},y_{ij}] - \sum_{i=1}^{m+n}[x_{ji},y_{ji}] = [b_{L_j},b_{R_j}], j = 1,2,3,\dots,n$$

these are the feasibility conditions of P. Hence $S = \{[x_{ij},y_{ij}]: [x_{ij},y_{ij}] \geq 0\}$ is feasible solution of P.

Splitting of Problem (P1) into Mid-value TP And Half-Width TP

Let a_{m_i} *and* $a_{w_i}; i = 1,2,\dots,m, m+1,\dots m+n$ be the mid-point and half-width of $\left[a_{L_i}+m_L, a_{R_i}+m_R\right], i = 1,2,3,\dots,m$ and $[m_L,m_R]; i = m+1, m+2,\dots,m+n$ respectively. Similarly b_{m_i} *and* $b_{w_i}; i = 1,2,\dots,m,m+1,\dots m+n$ be the mid-point and half-width of $\left[b_{L_i}+m_L, b_{R_i}+m_R\right], i = 1,2,3,\dots,m$ *and* $[m_L,m_R]; i = m+1, m+2,\dots,m+n$ respectively. Also m_{ij} is the mid-point and w_{ij} be the half width of $[x_{ij},y_{ij}]$ and $t_{m_{ij}}$ is the mid-point and $t_{w_{ij}}$ be the half width of $[t_{L_{ij}},t_{R_{ij}}]$

Then, the mid-value transportation problem of the problem P_1 can be defined as,

(M) $$\text{Minimize } Z = \max_{i,j}\left\{t_{m_{ij}}: m_{ij} > 0\right\}$$

Subject to,

$$\sum_{j=1}^{m+n} m_{ij} = a_{m_i}, i = 1,2,3,\dots,m, m+1,\dots,m+n \quad (7)$$

$$\sum_{i=1}^{m+n} m_{ij}, = b_{m_j}, j = 1,2,3,\dots,n, n+1,\dots,n+m \quad (8)$$

$$m_{ij}, \geq 0 ; i,j = 1,2,\dots m+n, i \neq j, i = 1,2,\dots m+n$$

$$a_{m_i} \geq 0 ; i = 1,2\dots m, m+1,\dots,m+n \text{ and } b_{m_j} \geq 0 ; j = 1,2,3,\dots,n,n+1,\dots,n+m$$

Similarly, the mid-value transportation problem of the problem P_1 can be defined as,

(W) $$\text{Minimize } Z = \max_{i,j}\left\{t_{w_{ij}}: w_{ij} > 0\right\}$$

Subject to,

$$\sum_{j=1}^{m+n} w_{ij} = a_{w_i}, i = 1,2,,\dots,m, m+1,\dots,m+n \quad (9)$$

$$\sum_{i=1}^{m+n} w_{ij} = b_{w_j}, j = 1,2, \dots n, n+1, \dots, m+n \quad (10)$$

$$a_{w_i} \geq 0 \, ; i = 1,2 \dots m, m+1, \dots, m+n \text{ and } b_{w_j} \geq 0 \, ; j = 1,2,3, \dots, n, n+1, \dots, n+m$$

Theorem : If the set $\{m'_{ij} : m'_{ij} \geq 0, \text{for all } i \text{ and } j\}$ be the feasible solution of the mid-value transportation problem (M) of P_1 and the set $\{w'_{ij} : w'_{ij} \geq 0, \text{for all } i \text{ and } j\}$ be any feasible solution of the half-width transportation problem (W) of P_1. Then, the set of intervals $\{ [m'_{ij} - w'_{ij}, m'_{ij}, + w'_{ij}] : \text{for all } i \text{ and } j\}$ is the feasible solution of the problem P_1, provided problem W has same basic cells as the problem M has.

Proof: Since $\{m'_{ij} : m'_{ij} \geq 0, \text{for all } i \text{ and } j\}$ and $\{w'_{ij} : w'_{ij} \geq 0, \text{for all } i \text{ and } j\}$ are feasible solution of the problems M and W respectively then

$$\sum_{j=1}^{m+n} m'_{ij} = a_{m_i}, i = 1,2,3, \dots, m, m+1, \dots, m+n$$

$$\sum_{i=1}^{m+n} m'_{ij} = b_{m_j}, j = 1,2,3, \dots, n, n+1, \dots, n+m$$

$$\sum_{j=1}^{m+n} w'_{ij} = a_{w_i}, i = 1,2,3, \dots, m, m+1, \dots, m+n$$

$$\sum_{i=1}^{m+n} w'_{ij} = b_{w_j}, j = 1,2,3 \dots n, n+1, \dots, m+n$$

Using the Result: if $m(A) = m_1 \text{ and } w(A) = w_1 \text{ then } A = [m_1 - w_1 \, , m_1 + w_1 \,]$, we can say

$$\sum_{j=1, j \neq i}^{m+n} [m'_{ij} - w'_{ij}, m'_{ij}, + w'_{ij}] = \left[a_{L_i} + m_L, a_{R_i} + m_R\right], i = 1,2,3, \dots, m$$

$$\sum_{j=1}^{m+n} [m'_{ij} - w'_{ij}, m'_{ij}, + w'_{ij}] = [m_L, m_R] \, , i = m+1, m+2, \dots m+n$$

$$\sum_{i=1}^{m+n} [m'_{ij} - w'_{ij}, m'_{ij}, + w'_{ij}] = [m_L, m_R] \, , \text{j} = 1,2, \dots m$$

$$\sum_{i=1}^{m+n} [m'_{ij} - w'_{ij}, m'_{ij}, + w'_{ij}] = \left[b_{L_j} + m_L, b_{R_j} + m_R\right], j = m+1, m+2, \dots m+n$$

Hence, $[m'_{ij} - w'_{ij}, m'_{ij}, + w'_{ij}]$ is the feasible solution of P_1.

5. Algorithm :

A new algorithm for finding the solution of interval integer transshipment problem (P) has been introduced. The proposed algorithm proceeds as follows.

Step 1. Construct the time minimizing interval transshipment table.

Step 2. Add buffer stock range to each transshipment point and convert the interval transshipment table into interval transportation problem (P_1) table.

Step 3. Split the problem into two independent transportation problems called, mid-value transportation problem (M) and half-width transportation problem (W) from the given problem P.

Step 4. Solve the problem M using shootout method [12]. Let $\{m'_{ij} : for \ all \ i \ and \ j\}$ be optimal solution of M.

Step 5. Solve the problem W such that the basic cells of the problem W must be same as that of the problem M. Let $\{w'_{ij} : for \ all \ i \ and \ j \}$ be solution of W.

Step 6. The feasible solution of the given problem (P_1) is,$[m'_{ij} - w'_{ij}, m'_{ij} + w'_{ij}]$.

Step 7. The feasible solution of the given problem (P) is obtain by applying the following change in the feasible solution of (P_1) ,If $\left[x_{ij}, y_{ij}\right] = \alpha$ and $\left[x_{ji}, y_{ji}\right] = \beta$; $\alpha \leq \beta$ then $\left[x_{ji}, y_{ji}\right] = \beta - \alpha \ provided \ \beta - \alpha \geq 0$, and for $\beta \leq \alpha, \left[x_{ij}, y_{ij}\right] = \alpha - \beta$ provide $\alpha - \beta \geq 0$. Note that if $[x_{ii}, y_{ii}] = \alpha$, this will give non-basic cell for transshipment problem, i.e. $[x_{ii}, y_{ii}] = 0$.

Find the corresponding time span with respect to this revised feasible solution.

6. Numerical Illustration :

An example to illustrated the step-wise working procedure of the proposed algorithm is as follows.

Example: The minimum supplies of a product from sources O_1, O_2 and O_3 are 20, 30 and 50 and the maximum supplies are 40, 50 and 60, respectively. The minimum demands for destination D_1, D_2, and D_3 are 30, 20, and 50 and the maximum demands are 60, 30 and 60, respectively. The shipping time ranges from each supply point to each demand point is given in Table 2. The objective of this problem is to create a strategy for moving the product from the supplier to the destination, while satisfying the supply and demand constraints in least possible time frame.

Table-2: *Time from supply to destinations*

	O_1	O_2	O_3	D_{3+1}	D_{3+2}	D_{3+3}
O_1	-	[1,3]	[3,5]	[12,18]	[16,20]	[10,16]
O_2	[1,3]	-	[1,4]	[18,24]	[10,14]	[12,16]
O_3	[3,5]	[1,4]	-	[14,20]	[20,26]	[18,26]
D_{3+1}	[12,18]	[18,24]	[14,20]	-	[2,4]	[1,2]
D_{3+2}	[16,20]	[10,14]	[20,26]	[2,4]	-	[2,6]
D_{3+3}	[10,16]	[12,16]	[18,26]	[1,2]	[2,6]	-

Solution:

Step1: The TMTsP-IC is given in table-3.

Table-3: *TMTsP-IC*

	O_1	O_2	O_3	D_{3+1}	D_{3+2}	D_{3+3}	Supply
O_1	-	[1,3]	[3,5]	[12,18]	[16,20]	[10,16]	[20,40]
O_2	[1,3]	-	[1,4]	[18,24]	[10,14]	[12,16]	[30,50]
O_3	[3,5]	[1,4]	-	[14,20]	[20,26]	[18,26]	[50,60]
D_{3+1}	[12,18]	[18,24]	[14,20]	-	[2,4]	[1,2]	-
D_{3+2}	[16,20]	[10,14]	[20,26]	[2,4]	-	[2,6]	-
D_{3+3}	[10,16]	[12,16]	[18,26]	[1,2]	[2,6]	-	-
Demand	-	-	-	[30,60]	[20,30]	[50,60]	[100,150]

Step 2: Buffer stock range $M = \sum_{i=1}^{3}[a_{L_i}, a_{R_i}] = \sum_{j=1}^{3}[b_{L_j}, b_{R_j}] = [100,150]$. The transshipment table-3 is converted into corresponding transportation table-4 by adding buffer stock range M= [100,150] to each transshipment point.

Table-4: *TMTP-IC*

	O_1	O_2	O_3	D_{3+1}	D_{3+2}	D_{3+3}	supply
O_1	-	[1,3]	[3,5]	[12,18]	[16,20]	[10,16]	[120,190]
O_2	[1,3]	-	[1,4]	[18,24]	[10,14]	[12,16]	[130,200]
O_3	[3,5]	[1,4]	-	[14,20]	[20,26]	[18,26]	[150,210]
D_{3+1}	[12,18]	[18,24]	[14,20]	-	[2,4]	[1,2]	[100,150]
D_{3+2}	[16,20]	[10,14]	[20,26]	[2,4]	-	[2,6]	[100,150]
D_{3+3}	[10,16]	[12,16]	[18,26]	[1,2]	[2,6]	-	[100,150]
Demand	[100,150]	[100,150]	[100,150]	[130,210]	[120,180]	[150,210]	[700,1050]

Step3: The table-4 is divided into two different crisp transportation tables: mid-value transportation table-5 and half-width transportation table-6.

Table-5: *mid value transportation table*

	O_1	O_2	O_3	D_{3+1}	D_{3+2}	D_{3+3}	supply
O_1	-	2	4	15	18	13	155

O_2	2	-	2.5	21	12	14	165
O_3	4	2.5	-	17	23	22	180
D_{3+1}	15	21	17	-	3	1.5	125
D_{3+2}	18	12	23	3	-	4	125
D_{3+3}	13	14	22	1.5	4	-	125
demand	125	125	125	170	150	180	875

Table-6: *half width transportation table*

	O_1	O_2	O_3	D_{3+1}	D_{3+2}	D_{3+3}	supply
O_1	-	1	1	3	2	3	35
O_2	1	-	1.5	3	2	2	35
O_3	1	1.5	-	3	3	4	30
D_{3+1}	3	3	3	-	1	0.5	25
D_{3+2}	2	2	3	1	-	2	25
D_{3+3}	3	2	4	0.5	2	-	25
demand	25	25	25	40	30	30	175

Step4: The optimal solution of the mid value transportation problem (table-5) is obtained by using shootout method and given in table -7. It can be obtained by other existing method for crisp TMTP.

Table-7: *Optimal solution of mid value transportation problem*

	O_1	O_2	O_3	D_{3+1}	D_{3+2}	D_{3+3}	supply
O_1	-[70]	2[85]	4	15	18	13	155
O_2	2	-[15]	2.5	21	12[150]	14	165
O_3	4[55]	2.5	-[125]	17	23	22	180
D_{3+1}	15	21	17	- [125]	3	1.5	125
D_{3+2}	18	12[25]	23	3	-	4[100]	125
D_{3+3}	13	14	22	1.5[45]	4	- [80]	125
demand	125	125	125	170	150	180	875

The solution of crisp mid-value TP is $m'_{11} = 70, m'_{12} = 85, m'_{22} = 15, m'_{25} = 150, m'_{31} = 55, m'_{33} = 125, m'_{44} = 125, m'_{52} = 25, m'_{56} = 100, m'_{62} = 25, m'_{66} = 80$.
Step 5. Half width-value transportation problem is solved in table-8 by giving assignment in feasible cells of table-7.

Table-8: *half mid width transshipment table*

	O_1	O_2	O_3	D_{3+1}	D_{3+2}	D_{3+3}	supply
O_1	[20]	1[15]	1	3	2	3	35
O_2	1	[5]	1.5	3	2[30]	2	35
O_3	1[5]	1.5	[25]	3	3	4	30
D_{3+1}	3	3	3	[25]	1	0.5	25
D_{3+2}	2	2[5]	3	1	-	2[20]	25
D_{3+3}	3	2	4	0.5[15]	2	[10]	25
demand	25	25	25	40	30	30	175

Solution of crisp half-width TP is $w'_{11} = 20, w'_{12} = 15, w'_{22} = 5, w'_{25} = 30, w'_{31} = 5, w'_{33} = 25, w'_{44} = 25, w'_{52} = 5, w'_{56} = 20, w'_{62} = 15, w'_{66} = 10$.
Step 6. The feasible solution of the transformed transportation problem is presented in table-9 and is obtained by combining the both solutions presented in table-7 and table-8.

Table-9: *solution of transformed transportation table*

	O_1	O_2	O_3	D_{3+1}	D_{3+2}	D_{3+3}	supply
O_1	[50,90]	[70,100]					[120,190]
O_2		[10,20]			[120,180]		[130,200]

O_3	[50,60]		[100,150]				[150,210]
D_{3+1}				[100,150]			[100,150]
D_{3+2}		[20,30]				[80,120]	[100,150]
D_{3+3}				[30,60]		[70,90]	[100,150]
demand	[100,150]	[100,150]	[100,150]	[130,210]	[120,180]	[150,210]	[700,1050]

$[x_{11}, y_{11}] = [50,90], [x_{12}, y_{12}] = [70,100], [x_{22}, y_{22}] = [10,20], [x_{25}, y_{25}] = [120,180], [x_{31}, y_{31}] = [50,60], [x_{33}, y_{33}] = [100,150], [x_{44}, y_{44}] = [100,150], [x_{52}, y_{52}] = [20,30], [x_{56}, y_{56}] = [80,120], [x_{64}, y_{64}] = [30,60], [x_{66}, y_{66}] = [70,90],$

Step7: The feasible solution of the original transshipment problem is as follow,
$[x_{12}, y_{12}] = [70,100], [x_{25}, y_{25}] = [120,180], [x_{31}, y_{31}] = [50,60], [x_{52}, y_{52}] = [20,30], [x_{56}, y_{56}] = [80,120], [x_{64}, y_{64}] = [30,60],$
corresponding time span is $\max_{i,j}\{[t_{ij}, s_{ij}]: [x_{ij}, y_{ij}] > 0\} = \max_{i,j}\{[1,3], [10,14], [3,5], [10,14], [2,6], [1,2]\}$
Whenever ordering the interval in lexicographic order, the time span is [10,14]. Likewise, when ordering by length, the time span is [10,14]. Additionally, when ordering intervals by the midpoint of the interval, the taken time span is [10,14].

7. CONCLUSION :

The present paper proposes a solution procedure of the time minimizing transshipment problem with interval constraints. A method called the mid-width method is established to find the basic feasible solution for a fully integer interval transshipment problem. For easy understanding and implementation, the method is summarized in an algorithm. This algorithm finds the feasible solution of the given problem in a finite number of steps. A numerical example illustrated the working procedure of our proposed algorithm and provided a feasible solution. The proposed method can be applied as a tool by decision makers to deal with real-life logistic problems involving interval parameters.

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